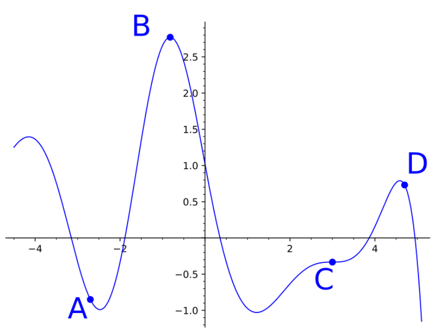
Final Exam Practice Problems

1. Below is the graph of a function $y = f(x)$. Fill in the chart with POS, NEG or 0 to indicate whether f, f' and f'' are positive, negative or zero at each of the indicated points A, B, C and D.



	f	f'	f''
A	NEG	NEG	POS
B	POS	0	NEG
C	NEG	0	0
D	POS	NEG	NEG

2. Find the following limits if they exist.

(a) $\lim_{x\to 0} \frac{\sqrt{5+x}-\sqrt{5}}{x} \times \frac{\sqrt{5+x}+\sqrt{5}}{\sqrt{5+x}+\sqrt{5}}$

MULTIPLY TOP & BOTTOM BY THE "RADICAL CONJUGATE" TO RATIONALIZE THE NUMERATOR

$$= \lim_{x\to 0} \frac{5+x-5}{x\left(\sqrt{5+x}+\sqrt{5}\right)}$$

$$\left(\left(\sqrt{a}-\sqrt{b}\right)\left(\sqrt{a}+\sqrt{b}\right) = \sqrt{a}^2 + \sqrt{a}\sqrt{b} - \sqrt{a}\sqrt{b} - \sqrt{b}^2 = a - b \right)$$

$$= \lim_{x\to 0} \frac{\cancel{x}}{\cancel{x}\left(\sqrt{5+x}+\sqrt{5}\right)}$$

PLUG IN $x=0$: $\lim_{x\to 0} \frac{1}{\sqrt{5+x}+\sqrt{5}} = \frac{1}{2\sqrt{5}}$

BOTH SHOW HOW TO FACTOR THE DIFFERENCE OF SQUARES

(b) $\lim_{x\to 1} \frac{x^2-1}{1-x^4} = \lim_{x\to 1} \frac{(x+1)(x-1)}{(1+x^2)(1-x^2)}$

$$\left(\text{FACTOR: } a^2-b^2 = (a+b)(a-b) \right)$$

$$= \lim_{x\to 1} \frac{(x+1)(x-1)}{(1+x^2)(1+x)(1-x)} = \lim_{x\to 1} \frac{\cancel{(x+1)}(-1)\cancel{(1-x)}}{(1+x^2)\cancel{(1+x)}\cancel{(1-x)}}$$

$$\left(a-b = -(b-a) \right)$$

PLUG IN $x=1$: $\lim_{x\to 1} \frac{-1}{1+x^2} = -\frac{1}{2}$

3. Compute the derivatives of the functions below.

(a) $f(x) = (3x^2 + x + 1)^{20}$

$y = u^{20}$, $u = 3x^2 + x + 1$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 20u^{19}(6x+1)$

$= \boxed{20(3x^2+x+1)^{19}(6x+1)}$

(b) $g(x) = \ln(1 + e^x)$

$y = \ln(u)$, $u = 1 + e^x$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{u} e^x = \frac{e^x}{1+e^x}$

(c) $h(x) = \frac{\sqrt{1+x}}{1-x}$ $= \frac{f(x)}{g(x)}$, $f(x) = (1+x)^{1/2}$ $\quad g(x) = 1-x$

$f'(x) = \frac{1}{2}(1+x)^{-1/2}$ $\quad g'(x) = -1$

$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} = \boxed{\frac{\frac{1}{2}(1+x)^{-1/2}(1-x) - (1+x)^{1/2}(-1)}{(1-x)^2}}$

(d) $j(x) = xe^{x^2}$ $= f(x)g(x)$, $f(x) = x$ $\quad g(x) = e^{x^2}$

$f'(x) = 1$ $\quad g'(x) = e^{x^2} \cdot 2x$

$j'(x) = f'(x)g(x) + f(x)g'(x) = \boxed{e^{x^2} + 2x^2 e^{x^2}}$

4. Compute the following indefinite integrals.

(a) $\int x^3 e^{x^4}\, dx$

$u = x^4$ (INNER FUNCTION)

$du = 4x^3\, dx$

$\frac{1}{4} du = x^3 dx$

$\rightsquigarrow \frac{1}{4}\int e^u\, du = \frac{1}{4} e^u + C$

$\rightsquigarrow \boxed{\frac{1}{4} e^{x^4} + C}$

(b) $\int \frac{(3x+4)^2}{x}\,dx = \int \frac{9x^2+24x+16}{x}\,dx = \int \frac{9x^2}{x}+\frac{24x}{x}+\frac{16}{x}\,dx$

$$= \int 9x + 24 + \frac{16}{x}\,dx = \boxed{\frac{9}{2}x^2 + 24x + 16\ln|x| + C}$$

5. Compute the following definite integrals.

(a) $\int_1^2 x^2\sqrt{x^3+1}\,dx$

$u = x^3 + 1$
$du = 3x^2\,dx$
$\frac{1}{3}\,du = x^2\,dx$

$$\int_{x=1}^{x=2} \longrightarrow \int_{u=1^3+1=2}^{u=2^3+1=9}$$

$$= \frac{1}{3}\int_2^9 u^{1/2}\,du = \frac{1}{3}\cdot\frac{2}{3}u^{3/2}\Big|_2^9 = \boxed{\frac{2}{9}\left(9^{3/2} - 2^{3/2}\right)}$$

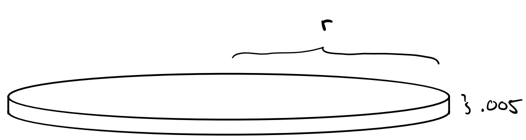
(b) $\int_2^{10} \frac{x}{3x^2-11}\,dx$

$u = 3x^2 - 11$
$du = 6x\,dx$
$\frac{1}{6}\,du = x\,dx$

WHEN $x = 10$, $u = 3\cdot 10^2 - 11 = 289$
WHEN $x = 2$, $u = 3\cdot 2^2 - 11 = 1$

$$\rightsquigarrow \frac{1}{6}\int_1^{289}\frac{1}{u}\,du = \frac{1}{6}\ln|u|\Big|_1^{289} = \frac{1}{6}\ln 289 - \frac{1}{6}\underbrace{\ln 1}_{0}$$

6. An offshore oil well is leaking oil onto the ocean surface, forming a circular oil slick about 0.005 meters thick. If the radius of the slick is r meters, then the volume of the oil spilled is $0.005\pi r^2$ cubic meters. Suppose the oil is leaking at a constant rate of 20 cubic meters per hour. Find the rate at which the radius of the oil slick is increasing at a time when the radius of the oil slick is 50 meters.



GIVEN $\frac{dV}{dt} = 20\ m^3/h$

FIND $\frac{dr}{dt}$ WHEN $r = 50\ m$

① FIRST RELATE V & r

$V = .005\pi r^2 \xrightarrow[\text{DIFF.}]{\text{IMP.}}$

② USE IMPLICIT DIFFERENTIATION TO RELATE $\frac{dV}{dt}$ & $\frac{dr}{dt}$.

$$\frac{d}{dt}[V] = \frac{d}{dt}[.005\pi r^2]$$

$$\frac{dV}{dt} = .005\pi\cdot 2r\frac{dr}{dt} = .01\pi r\frac{dr}{dt}$$

③ SOLVE ALGEBRAICALLY FOR $\frac{dr}{dt}$

$$\frac{dr}{dt} = \frac{\frac{dV}{dt}}{.01\pi r} = \frac{20}{.01\pi(50)} = \frac{40}{\pi}$$

④ NOW PLUG IN $r = 50$, $\frac{dV}{dt} = 20$

7. Let

$$f(x) = 3 - \frac{2}{1+x}.$$

Use the definition of the derivative as a limit to find $f'(3)$.

WE HAVE TWO EQUIVALENT DEFINITIONS: (1) $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

(2) $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

USE EITHER ONE.

(1) $$f'(3) = \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3} = \lim_{x \to 3} \frac{\cancel{3} - \frac{2}{1+x} - \left(\cancel{3} - \frac{2}{1+3}\right)}{x - 3}$$

$$= \lim_{x \to 3} \frac{\frac{1}{2} - \frac{2}{1+x}}{x - 3} = \lim_{x \to 3} \frac{1}{x-3} \left(\frac{1 + x - 4}{2(1+x)} \right)$$ COMMON DENOMINATORS

$$= \lim_{x \to 3} \frac{1}{\cancel{x-3}} \cdot \frac{\cancel{x-3}}{2(1+x)} = \lim_{x \to 3} \frac{1}{2(1+x)} = \frac{1}{2(1+3)} = \frac{1}{8}$$

(2) $$f'(3) = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h \to 0} \frac{\cancel{3} - \frac{2}{1+3+h} - \left(\cancel{3} - \frac{2}{1+3}\right)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \left(\frac{1}{2} - \frac{2}{4+h} \right) = \lim_{h \to 0} \frac{1}{h} \left(\frac{4 + h - 4}{2(4+h)} \right)$$ COMMON DENOMINATORS

$$= \lim_{h \to 0} \frac{1}{\cancel{h}} \cdot \frac{\cancel{h}}{2(4+h)} = \lim_{h \to 0} \frac{1}{2(4+h)} = \frac{1}{2(4+0)} = \frac{1}{8}$$

8. A bacteria culture starts with 500 bacteria and grows at a rate proportional to its size. After 3 hours there are 8000 bacteria.

(a) Find an expression for the population after t hours.

(b) After how many hours will the population reach 100,000?

(a) EXPONENTIAL GROWTH / DECAY MODEL: $P(t) = Ce^{kt}$

GIVEN $P(0) = 500 \Rightarrow \underbrace{Ce^{0k}}_{1} = 500 \Rightarrow C = 500$

GIVEN $P(3) = 8000 \Rightarrow \frac{500e^{3k}}{500} = \frac{8000}{500} \Rightarrow e^{3k} = 16$

NOW $e^{3k} = 16$

(EITHER WAY)

$$\left(e^{3k}\right)^{1/3} = (16)^{1/3}$$

$$e^{k} = 16^{1/3}$$

$$\Rightarrow P(t) = 500\left(16^{1/3}\right)^{t}$$

$$\boxed{P(t) = 500 \cdot 16^{t/3}}$$

$$\ln\left(e^{3k}\right) = \ln(16)$$

$$3k = \ln(16)$$

$$k = \tfrac{1}{3}\ln(16)$$

$$\boxed{P(t) = 500\, e^{\frac{1}{3}\ln(16)\, t}}$$

SAME

FYI, THIS NUMBER $k = \frac{1}{3}\ln(16) \approx .92$ IS CALLED THE RELATIVE GROWTH RATE & SHOWS THAT THE POPULATION IS ALWAYS GROWING AT A RATE OF 92% PER HOUR.

NOTE THAT THE SAME FORMULA COULD BE USED TO CALCULATE THE VALUE OF A \$500 INVESTMENT EARNING ≈ 92% INTEREST COMPOUNDED CONTINUOUSLY.

(b) SOLVE FOR t: $P(t) = \dfrac{500 \cdot 16^{t/3}}{500} = \dfrac{100{,}000}{500}$

$$\ln\left(16^{t/3}\right) = \ln(200)$$

$$\frac{t}{3}\ln(16) = \ln(200)$$

$$t = \boxed{\frac{3\ln(200)}{\ln(16)} \text{ HOURS}}$$

9. A curve is defined by the equation

$$x^2 - y^2 + 4x + 8 = 0.$$

Give an equation for the tangent line to the curve at the point $(1, 4)$.

THE EQUATION WILL HAVE THE FORM $y - 4 = m(x-1)$

WHERE $m = \dfrac{dy}{dx}$ EVALUATED @ $(1,4)$ i.e. $\left.\dfrac{dy}{dx}\right|_{(1,4)}$

(THE POINT-SLOPE EQUATION OF THE LINE THROUGH (a,b) WITH SLOPE m IS $y - b = m(x-a)$.)

WE FIND $\frac{dy}{dx}$ BY IMPLICIT DIFFERENTIATION:

$$\frac{d}{dx}\left[x^2 - y^2 + 4x + 8\right] = \frac{d}{dx}[0]$$

$$2x - 2y\frac{dy}{dx} + 4 = 0 \quad \Rightarrow \quad -2y\frac{dy}{dx} = -2x - 4$$

$$\therefore \quad \frac{dy}{dx} = \frac{2x+4}{2y} \qquad \text{AND SO} \qquad m = \left.\frac{dy}{dx}\right|_{(1,4)} = \frac{2(1)+4}{2(4)} = \frac{6}{8} = \frac{3}{4}$$

NOW THE POINT-SLOPE EQUATION OF THE LINE IS

$$\boxed{y - 4 = \frac{3}{4}(x-1)} \qquad \left(\text{IN SLOPE-INTERCEPT FORM THIS IS } y = \frac{3}{4}x + \frac{13}{4}\right)$$

10. Find the absolute maximum and minimum value for the function $f(x) = 9x - 3x^2 - x^3$ on the interval $[-4, 2]$.

THE ABSOLUTE MAX/MIN OF A CONTINUOUS FUNCTION f ON A CLOSED INTERVAL $[a,b]$ MUST OCCUR AT EITHER A CRITICAL NUMBER OF f IN THE INTERVAL $[a,b]$ OR AT a OR b.

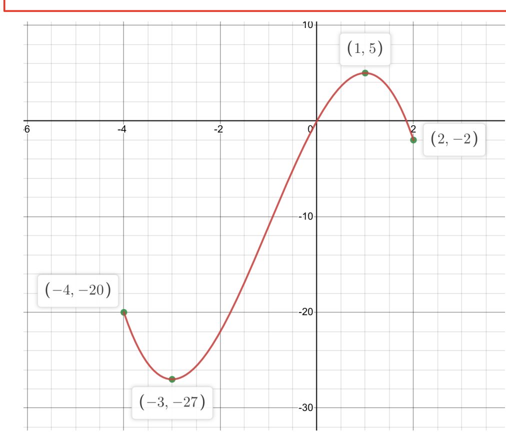
CRIT. #(S): $f'(x) = -3x^2 - 6x + 9 = -3(x^2 + 2x - 3) = 3(x+3)(x-1)$

$$f'(x) = 0 \Rightarrow 3(x+3)(x-1) = 0$$

$$\Rightarrow x = -3 \text{ OR } x = 1 \qquad (\text{BOTH CRIT. \#'S ARE IN } [-4,2])$$

x	f(x)
-4	-20
-3	-27
1	5
2	-2

$$\boxed{\begin{array}{l}\text{ABSOLUTE MINIMUM VALUE } f(-3) = -27 \\ \text{ABSOLUTE MAXIMUM VALUE } f(1) = 5\end{array}}$$



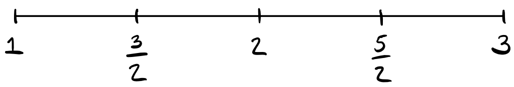
(GRAPH FYI)

11. Use a Riemann sum with 4 subintervals and left endpoints (L_4) to estimate the integral

$$\int_1^3 9^x \, dx.$$

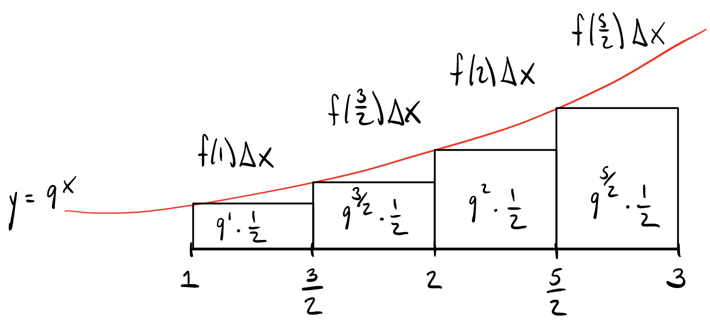
① SPLIT THE INTERVAL $[1,3]$ INTO $n=4$ SUBINTERVALS ALL OF EQUAL LENGTH Δx.

$$\Delta x = \frac{b-a}{n} = \frac{3-1}{4} = \frac{1}{2}$$



② FOR EACH SUBINTERVAL $[1,\frac{3}{2}]$, $[\frac{3}{2},2]$, $[2,\frac{5}{2}]$, $[\frac{5}{2},3]$,

PLUG THE LEFT ENDPOINT INTO $f(x) = 9^x$ AND MULTIPLY BY $\Delta x = \frac{1}{2}$.



EACH PRODUCT IS THE AREA OF A RECTANGLE WHOSE BASE IS ALONG THE X-AXIS & TOP LEFT CORNER IS ON THE CURVE $y = f(x)$.

③ THE RIEMANN SUM IS THE SUM OF THESE PRODUCTS.

$$L_4 = 9\cdot\frac{1}{2} + 9^{3/2}\cdot\frac{1}{2} + 9^2\cdot\frac{1}{2} + 9^{5/2}\cdot\frac{1}{2}$$

$$= \left(9 + (9^{1/2})^3 + 9^2 + (9^{1/2})^5\right)\cdot\frac{1}{2}$$

$$= (9 + 27 + 81 + 243)\,\frac{1}{2} = 360\cdot\frac{1}{2} = 180$$