Vector Calc Review

John Adamski

May 23, 2017

1 Line Integrals

Given a differentiable curve C parametrized by

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle, \quad a \leqslant t \leqslant b$$

the arc length of C is

$$\int_C ds = \int_a^b |\vec{r}\prime(t)| \, dt.$$

Given a scalar function (density) $f(x, y, z)$, the line integral of f along C (mass) is

$$\int_C f \, ds = \int_a^b f(\vec{r}(t)) |\vec{r}\prime(t)| \, dt.$$

Given a vector (force) field $\vec{F}(x, y, z)$ and an orientation (direction) for C, the line integral of $\vec{F}$ along C (work) is

$$\int_C \vec{F} \cdot \vec{T} \, ds = \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{\vec{r}\prime(t)}{|\vec{r}\prime(t)|} |\vec{r}\prime(t)| \, dt = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\prime(t) \, dt = \int_C \vec{F} \cdot d\vec{r}. \tag{1}$$

Remark. Note that

$$\vec{r}\prime(t) \, dt = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle dt = \langle dx, dy, dz \rangle.$$

Thus, letting $\vec{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle$, equation (1) can also be written

$$\int_C P \, dx + Q \, dy + R \, dz.$$

2 Surface Integrals

Given a surface S parametrized by

$$\vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle, \quad (u, v) \in D$$

the surface area of S is

$$\iint_S dS = \iint_D |\vec{r}_u \times \vec{r}_v| \, dA.$$

In the special case that S is the graph $z = g(x, y)$ then a simple parametrization of S is

$$\vec{r}(x, y) = \langle x, y, g(x, y) \rangle, \quad (x, y) \in D$$

and its surface area is

$$\iint_S dS = \iint_D |\langle -g_x(x, y), -g_y(x, y), 1 \rangle| \, dA = \iint_D \sqrt{g_x^2 + g_y^2 + 1} \, dA.$$

Given a scalar function (density) $f(x, y, z)$, the surface integral of f over S (mass) is

$$\iint_S f \, dS = \iint_D f(\vec{r}(u, v)) |\vec{r}_u \times \vec{r}_v| \, dA.$$

Given a vector (force) field $\vec{F}(x, y, z)$ and an orientation for S, the surface integral of $\vec{F}$ over S (flux) is

$$\iint_S \vec{F} \cdot \vec{n} \, dS = \iint_D \vec{F}(\vec{r}(u, v)) \cdot \frac{\pm \vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} |\vec{r}_u \times \vec{r}_v| \, dA = \iint_D \vec{F}(\vec{r}(u, v)) \cdot (\pm \vec{r}_u \times \vec{r}_v) \, dA. \tag{2}$$

The $\pm$ is determined by the orientation of the surface S.

Remark. Letting $d\vec{S} = \vec{n} \, dS$, equation (2) can also be written

$$\iint_S \vec{F} \cdot d\vec{S}.$$

3 Theorems

Theorem 3.1 (Fundamental Theorem for Line Integrals)**.** Let C be a smooth curve given by the vector function $\vec{r}(t)$, $a \leqslant t \leqslant b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)).$$

Theorem 3.2 (Conervative vector field)**.** Given a vector field $\vec{F}$, there exists a scalar function f such that $\vec{F} = \nabla f$ if and only if $\vec{F}$ is *conservative.* In two dimensions, $\vec{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$ is conservative if

1. $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$, and

2. The domain of $\vec{F}$ is *simply connected*. That is, the domain of $\vec{F}$ has "no holes".

In three dimensions, $\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$ is conservative if

1. $\operatorname{curl} \vec{F} = \vec{0}$, and
2. The domain of $\vec{F}$ is $\mathbb{R}^3$.

Thus the content of this theorem is to give a way of knowing when a vector field $\vec{F}$ has a potential function f.

Theorem 3.3 (Green's Theorem)**.** Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dA.$$

Theorem 3.4 (Stokes' Theorem)**.** Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\vec{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S}.$$

Theorem 3.5 (Divergence Theorem)**.** Let E be a simple solid region and let S be the boundary surface of E, given with positive (outward) orientation. Let $\vec{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\,dV.$$