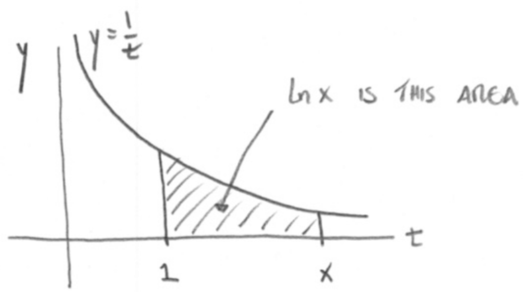
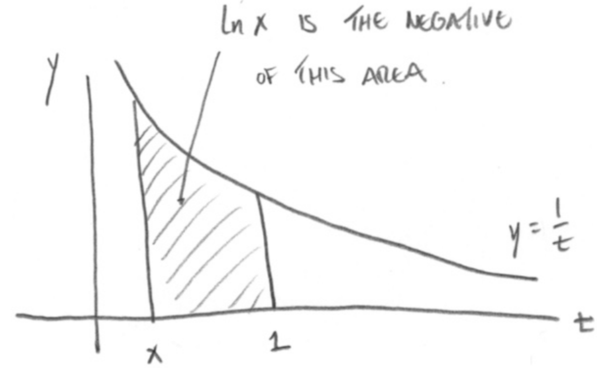
§7.1 THE LOGARITHM DEFINED AS AN INTEGRAL

DEF: THE NATURAL LOGARITHM

$$\ln x = \int_1^x \frac{1}{t}\, dt$$





- $\ln 1 = 0$

- $\mathrm{Dom}(\ln) = (0, \infty)$

- FTC $\Rightarrow$ $\dfrac{d}{dx} \ln x = \dfrac{1}{x}$, $x > 0$ (DIFFERENTIABLE, CONTINUOUS)

$$\frac{d}{dx} \ln(f(x)) = \frac{1}{f(x)} f'(x), \quad f(x) > 0$$

$$\hookrightarrow \frac{d}{dx} \ln|x| = \begin{cases} \frac{d}{dx} \ln x, & x > 0 \\ \frac{d}{dx} \ln(-x), & x < 0 \end{cases} = \frac{1}{x}, \quad x \neq 0$$

- <u>NOW</u> WE CAN PROVE LOG RULES:

e.g. $\dfrac{d}{dx} \ln(ax) = \dfrac{1}{ax} \cdot a = \dfrac{1}{x} = \dfrac{d}{dx} \ln x$

$\therefore \ln(ax) = \ln x + C$ (COR. 2, §4.2)

LET $x = 1 \Rightarrow C = \ln a$ $\square$

ex. $\displaystyle\int \frac{\ln(\ln x)}{x \ln x}\,dx$ $\qquad u = \ln x$, $\quad du = \frac{1}{x}\,dx$

$\displaystyle\int \frac{\ln u}{u}\,du$ $\qquad w = \ln u$, $\quad dw = \frac{1}{u}\,du$

$$\int w\,dw = \frac{1}{2}w^2 + C = \frac{1}{2}(\ln u)^2 + C$$

$$= \frac{1}{2}\left(\ln(\ln x)\right)^2 + C$$

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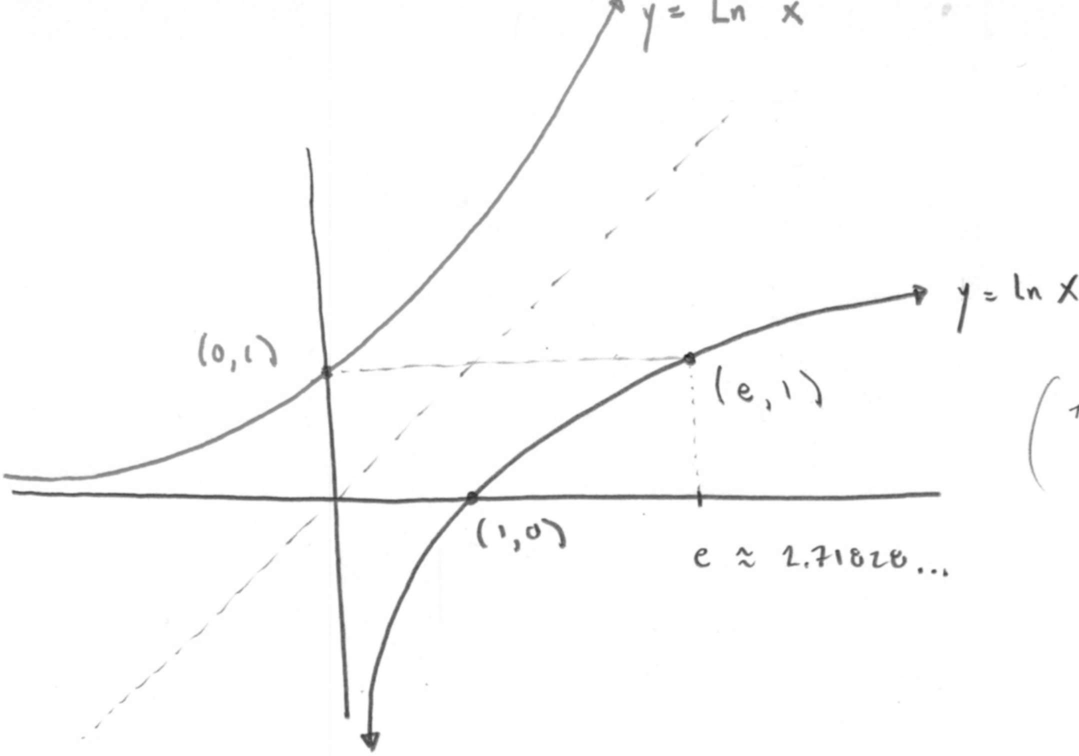
ex. $\displaystyle\int \frac{\sec y \tan y}{2 + \sec y}\,dy$ $\qquad u = 2 + \sec y$, $\quad du = \sec y \tan y\,dy$ $\quad \ldots$

DON'T FORGET ABS. VAL!

$\ln|\quad|$

- NOTE THAT $\ln$ IS INCREASING (POSITIVE 1ST DERIV.), CONCAVE DOWN (NEG 2nd DERIV.)
- $\ln(2^n) = n\ln(2) > \frac{n}{2} \longrightarrow \infty$ AS $n \to \infty$

$$\therefore \lim_{x\to\infty} \ln x = \infty, \quad \lim_{x\to 0^+} \ln x = \lim_{x\to\infty} \ln\frac{1}{x} = -\infty$$



(THIS IS A <u>DEFINITION</u> FOR e)

WHAT IS $\ln^{-1}$? NOTE: $\ln e^r = r \ln e = r$

$$\Rightarrow \cancel{\ln^{-1}} (\cancel{\ln} e^r) = \ln^{-1} r$$

$$e^r = \ln^{-1} r$$

$$\boxed{\begin{array}{lll} f(x) = \ln x & \multirow{2}{*}{\Big\} \text{ INVERSE FUNCTIONS}} & \ln(e^x) = x, \quad \text{FOR ALL } x \\ g(x) = e^x & & e^{\ln x} = x, \quad x > 0 \end{array}}$$

DERIV. OF e^x : LET $y = e^x$. FIND y'.

$$\ln y = x$$

$$\frac{1}{y} y' = 1 \Rightarrow y' = y = e^x$$

MORE GENERALLY, $\frac{d}{dx} e^{f(x)} = e^{f(x)} f'(x)$

$$\left(\int a^x \, dx = \frac{1}{\ln a} a^x + C \right)$$

IN PARTICULAR $\frac{d}{dx} a^x = \frac{d}{dx} e^{x \ln a} = e^{x \ln a} \ln a = a^x \ln a$

ex. $\displaystyle\int_2^4 x^{2x}(1+\ln x)\,dx$

$u = 2x\ln x$

$du = 2(\ln x + 1)\,dx$

$$\frac{1}{2}\int_{4\ln 2}^{8\ln 4} e^u\,du = \ldots$$

•) NOW WE CAN PROVE RULES FOR EXPONENTS

e.g. LET $a = e^x$, $b = e^y$ ($x = \ln a$, $y = \ln b$)

$$x + y = \ln a + \ln b = \ln(ab)$$

$$\Rightarrow \quad e^{(x+y)} = e^{\ln(ab)} = ab = e^x e^y$$

$\log_a x$ IS DEFINED TO BE THE INVERSE OF a^x

AND $\log_a x = \dfrac{\ln x}{\ln a}$

$$\left(\begin{aligned} &\text{SET } y = a^x \\ &\ln y = x \ln a \\ &x = \frac{\ln y}{\ln a} \end{aligned} \right)$$

$a^{\log_a x} = x$, $x > 0$

$\log_a(a^x) = x$, FOR ALL x

$$\frac{d}{dx} \log_a x = \frac{1}{x \ln a}, \qquad \frac{d}{dx} \log_a f(x) = \frac{f'(x)}{f(x) \ln a}$$

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ex. $$\int_1^e \frac{2 \ln 10 \log_{10} x}{x} \, dx = 2 \int_1^e \frac{\ln x}{x} \, dx = (\ln x)^2 \Big|_1^e = 1$$

ex. $$\int \frac{dx}{x \log_{10} x}$$

ex. $$\int \frac{dx}{x (\log_8 x)^2}$$

THE QUESTION ABOUT WHAT IF WE CHANGED THE CONSTANT 1 TO SOME OTHER #

IN THE DEFINITION

$$\boxed{\ln x := \int_{1}^{x} \frac{1}{t}\,dt}$$

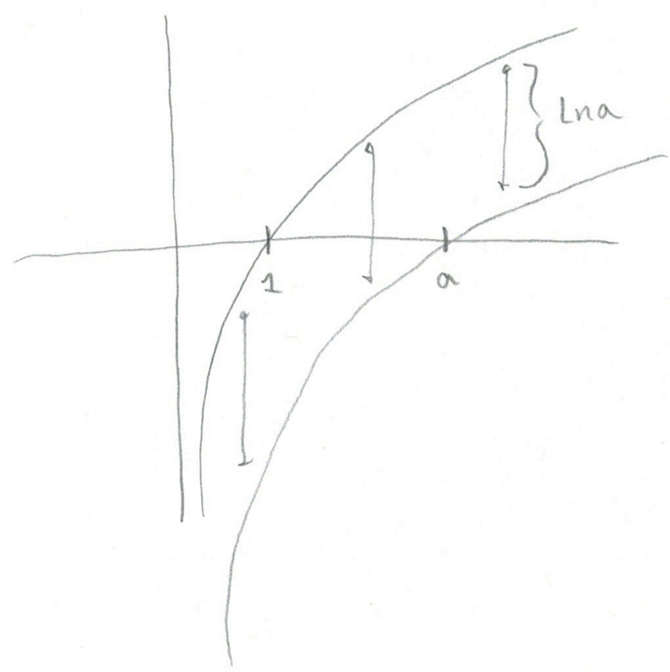
WHY?

SAY $\quad \text{Lm}\, x := \displaystyle\int_{a}^{x} \frac{1}{t}\,dt \quad , \quad a > 0$

$$= \int_{a}^{1} \frac{1}{t}\,dt + \int_{1}^{x} \frac{1}{t}\,dt$$

$$= -\int_{1}^{a} \frac{1}{t}\,dt + \int_{1}^{x} \frac{1}{t}\,dt$$

$$= -\ln a + \ln x$$



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