

Def: Let $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$.

$$\text{Then } \vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

Note: CROSS PRODUCT IS ONLY DEFINED FOR 3D VECTORS.

TRICK FOR REMEMBERING:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \hat{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \hat{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

↑
DETERMINANTS

e.g. COMPUTE $\langle 5, -8, 5 \rangle \times \langle -3, 1, -4 \rangle$

THEM: $\vec{a} \times \vec{b}$ (THE VECTOR) IS ORTHOGONAL TO BOTH $\vec{a}$ & $\vec{b}$.

Proof: $(\vec{a} \times \vec{b}) \cdot \vec{a} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle \cdot \langle a_1, a_2, a_3 \rangle$

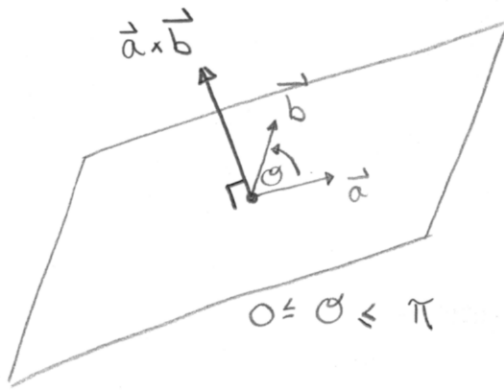
$$= a_1 a_2 b_3 - a_1 a_3 b_2 + a_2 a_3 b_1 - a_2 a_1 b_3 + a_3 a_1 b_2 - a_3 a_2 b_1$$

$$= 0.$$

& SIMILARLY, $(\vec{a} \times \vec{b}) \cdot \vec{b} = 0. \quad \square$

DIRECTION: $\vec{a} \times \vec{b}$ POINTS PERPENDICULAR TO PLANE CONTAINING $\vec{a}, \vec{b}$

ACCORDING TO RIGHT HAND RULE



FINGERS CURL FROM $\vec{a}$ TO $\vec{b}$.
THUMB POINTS IN DIRECTION OF $\vec{a} \times \vec{b}$.

MAGNITUDE: IF θ IS ANGLE BETWEEN $\vec{a}$ & $\vec{b}$ ($0 \leq \theta \leq \pi$)
(THM) THEN $|\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin \theta$.

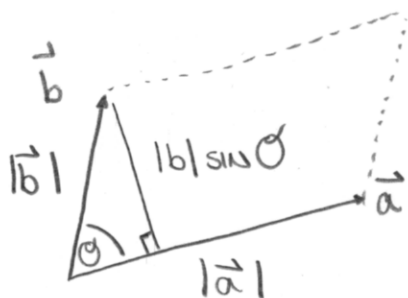
PROOF: $|\vec{a} \times \vec{b}|^2 = (a_2 b_3 - a_3 b_2)^2 + (a_3 b_1 - a_1 b_3)^2 + (a_1 b_2 - a_2 b_1)^2$
 $= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1 b_1 + a_2 b_2 + a_3 b_3)^2$
 $= |\vec{a}|^2 |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2$
 $= |\vec{a}|^2 |\vec{b}|^2 - |\vec{a}|^2 |\vec{b}|^2 \cos^2 \theta$
 $= |\vec{a}|^2 |\vec{b}|^2 (1 - \cos^2 \theta)$
 $= |\vec{a}|^2 |\vec{b}|^2 \sin^2 \theta$

$$\Rightarrow |\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| |\sin \theta| = |\vec{a}| |\vec{b}| \sin \theta \quad \text{SINCE } 0 \leq \theta \leq \pi \quad \square$$

Cor: IF $\vec{a}, \vec{b}$ NON ZERO, THEN $\vec{a} \parallel \vec{b} \Leftrightarrow \vec{a} \times \vec{b} = \vec{0}$.

PROOF: ...

FACT: $|\vec{a} \times \vec{b}| = \text{AREA OF PARALLELOGRAM DETERMINED BY } \vec{a} \text{ \& } \vec{b}$



e.g. FIND A VECTOR $\perp$ TO PLANE THROUGH THE 3 POINTS
 $(1, 2, 1)$, $(1, 4, 2)$, $(3, -3, -2)$.

e.g. FIND AREA OF TRIANGLE WITH SAME VERTICES. $\rightarrow$

NOTE: $\hat{i} \times \hat{j} = \hat{k}$

$$\hat{j} \times \hat{i} = -\hat{k}$$

$$\hat{j} \times \hat{k} = \hat{i}$$

$$\hat{k} \times \hat{j} = -\hat{i}$$

$$\hat{k} \times \hat{i} = \hat{j}$$

$$\hat{i} \times \hat{k} = -\hat{j}$$

- NOT COMMUTATIVE!

NOTE: $\hat{i} \times (\hat{i} \times \hat{j}) = \hat{i} \times \hat{k} = -\hat{j}$

$$(\hat{i} \times \hat{i}) \times \hat{j} = \vec{0} \times \hat{j} = \vec{0}$$

- NOT ASSOCIATIVE!

PROPERTIES OF CROSS PRODUCT

Let $\vec{a}, \vec{b}, \vec{c}$ be vectors & c be a scalar.

(i) $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$ [ANTI-COMMUTATIVE]

(ii) $(c\vec{a}) \times \vec{b} = \vec{a} \times (c\vec{b}) = c(\vec{a} \times \vec{b})$

(iii) $\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$ [LEFT-DISTRIBUTION]

(iv) $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$ [RIGHT-DISTRIBUTION]

(v) $\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{c} \cdot (\vec{a} \times \vec{b}) = \vec{b} \cdot (\vec{c} \times \vec{a})$

"SCALAR TRIPLE PRODUCT"

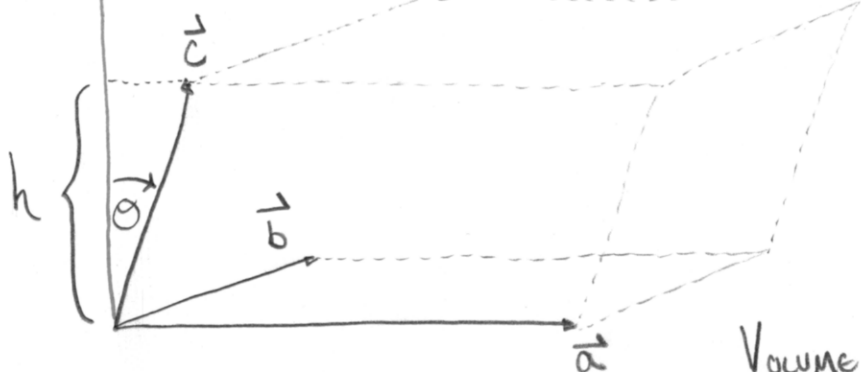
(vi) $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{a} \times \vec{c}) - \vec{c} \cdot (\vec{a} \times \vec{b})$

MEMORIC: BAC - CAB

"VECTOR TRIPLE PRODUCT"

SCALAR TRIPLE PRODUCT

$\vec{a} \times \vec{b}$



PARALLELOPIPED DETERMINED BY $\vec{a}, \vec{b}, \vec{c}$.

VOLUME $V = Bh$

AREA OF BASE $B = |\vec{a} \times \vec{b}|$

HEIGHT $h = |\vec{c}| |\cos \theta|$

$= |\vec{a} \times \vec{b}| |\vec{c}| |\cos \theta|$

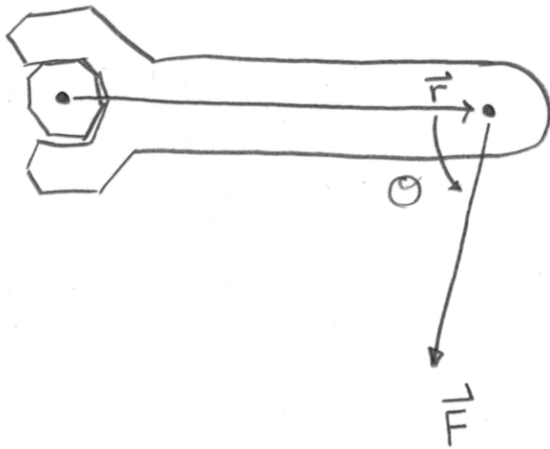
$= |(\vec{a} \times \vec{b}) \cdot \vec{c}|$

Note: Vectors $\vec{a}, \vec{b}, \vec{c}$ ARE COPLANAR

$\Leftrightarrow$ THE VOLUME OF THEIR CORRESPONDING
PARALLELOPIPED IS 0.

e.g. SHOW THAT $\langle -2, 1, -2 \rangle, \langle 7, -1, 8 \rangle, \langle 5, 0, 6 \rangle$
ARE COPLANAR.

TORQUE



$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$|\vec{\tau}| = |\vec{r} \times \vec{F}| = |\vec{r}| |\vec{F}| \sin \theta$$

§12.4 cross product

2 ways to multiply vectors $\vec{a} = \langle x_1, y_1, z_1 \rangle$

$$\vec{b} = \langle x_2, y_2, z_2 \rangle$$

(1) Dot Product / scalar product

$$\vec{a} \cdot \vec{b} = x_1 x_2 + y_1 y_2 + z_1 z_2 = |\vec{a}| |\vec{b}| \cos \theta$$

(2) cross product / vector product.

$$\vec{a} \times \vec{b} = \langle y_1 z_2 - z_1 y_2, z_1 x_2 - x_1 z_2, x_1 y_2 - y_1 x_2 \rangle$$

↑ Note: cross product is only defined for 3D vectors.

determinant of 2×2 matrix = $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$\vec{a} \times \vec{b}$ is determinant of 3×3 matrix

$$\begin{matrix} \vec{a} \rightarrow \\ \vec{b} \rightarrow \end{matrix} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

$$\begin{aligned} \vec{a} \times \vec{b} &= \hat{i} \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} - \hat{j} \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} + \hat{k} \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \\ &= \hat{i} (y_1 z_2 - z_1 y_2) - \hat{j} (x_1 z_2 - z_1 x_2) + \hat{k} (x_1 y_2 - y_1 x_2) \\ &= \langle y_1 z_2 - z_1 y_2, z_1 x_2 - x_1 z_2, x_1 y_2 - y_1 x_2 \rangle \end{aligned}$$

Note: $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$ ANTI-COMMUTATIVE

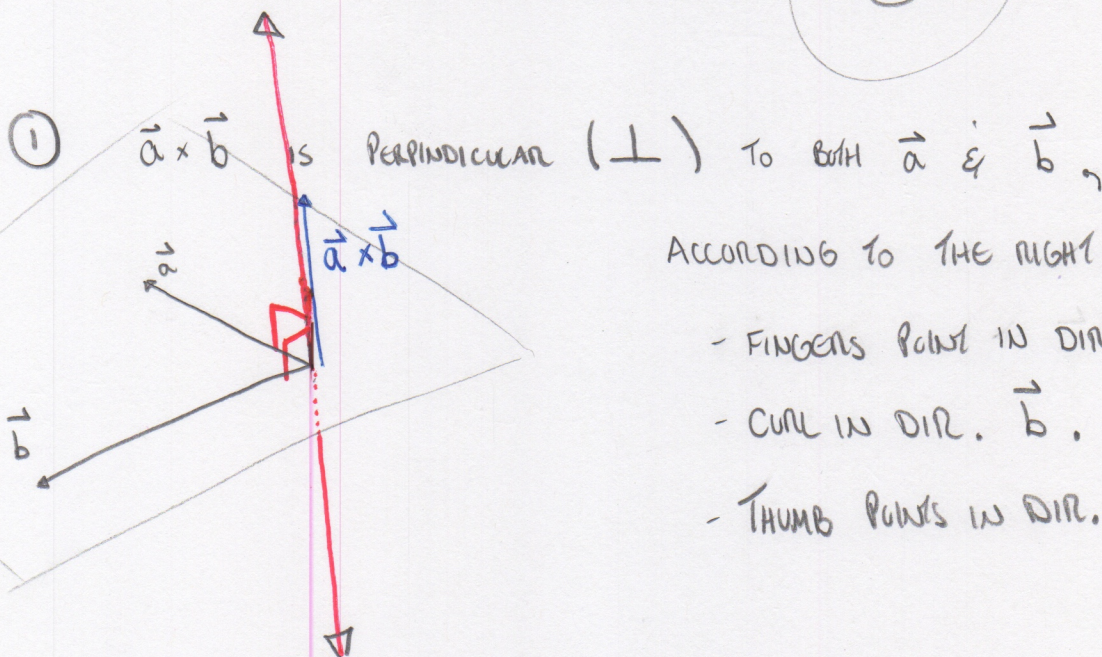
ex. $\langle 1, 2, 0 \rangle \times \langle -4, 0, -3 \rangle$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 0 \\ -4 & 0 & -3 \end{vmatrix} = \hat{i} \begin{vmatrix} 2 & 0 \\ 0 & -3 \end{vmatrix} - \hat{j} \begin{vmatrix} 1 & 0 \\ -4 & -3 \end{vmatrix} + \hat{k} \begin{vmatrix} 1 & 2 \\ -4 & 0 \end{vmatrix}$$

$$= \hat{i}(-6-0) - \hat{j}(-3-0) + \hat{k}(0-(-8))$$

$$= \langle -6, 3, 8 \rangle$$

$\vec{a} \times \vec{b}$ is vector HAS DIRECTION (1) & MAGNITUDE (2)



ACCORDING TO THE RIGHT HAND RULE:

- FINGERS POINT IN DIR. $\vec{a}$
- CURLED IN DIR. $\vec{b}$.
- THUMB POINTS IN DIR. $\vec{a} \times \vec{b}$

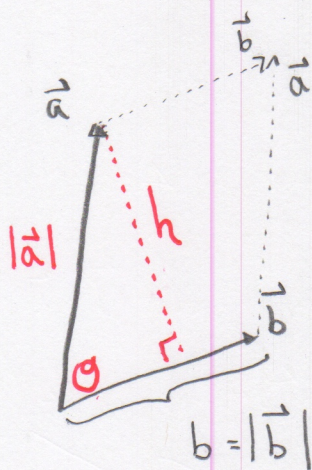
② MAGNITUDE: IF θ IS ANGLE BETWEEN $\vec{a}$ & $\vec{b}$ ($0 \leq \theta \leq \pi$)

THEN $|\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin \theta$

~~$(\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta)$~~

(BOOK CONTAINS PROOF).

NOTE: $|\vec{a} \times \vec{b}|$ IS AREA OF PARALLELOGRAM
WITH SIDES PARALLEL TO $\vec{a}$ & $\vec{b}$.



$$\text{Area} = bh = |\vec{b}| |\vec{a}| \sin \theta$$

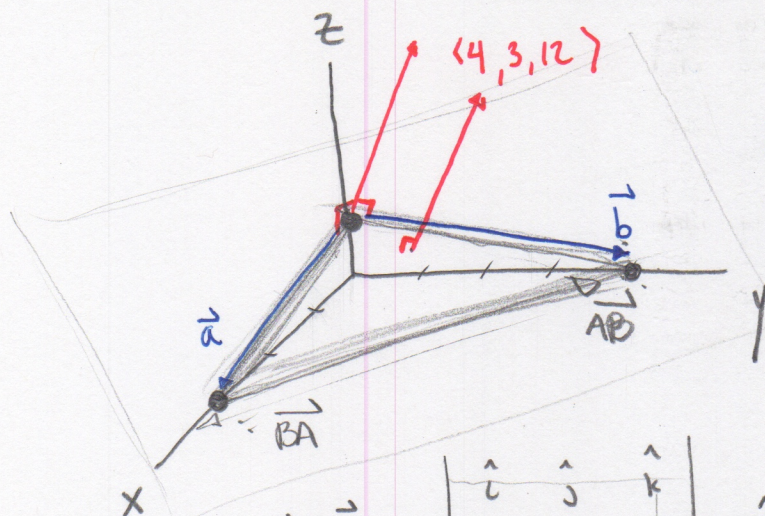
SINCE $0 \leq \theta \leq \pi$,

$$0 \leq \sin \theta$$

$$\text{Area} = |\vec{a}| |\vec{b}| \sin \theta = |\vec{a} \times \vec{b}|$$

ex. FIND VECTOR $\perp$ TO PLANE CONTAINING

$$\underbrace{(3, 0, 0)}_A, \underbrace{(0, 4, 0)}_B, \underbrace{(0, 0, 1)}_C$$



$$\vec{a} = \vec{CA} = \langle 3, 0, -1 \rangle$$

$$\vec{b} = \vec{CB} = \langle 0, 4, -1 \rangle$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 0 & -1 \\ 0 & 4 & -1 \end{vmatrix} = \hat{i} \begin{vmatrix} 0 & -1 \\ 4 & -1 \end{vmatrix} - \hat{j} \begin{vmatrix} 3 & -1 \\ 0 & -1 \end{vmatrix} + \hat{k} \begin{vmatrix} 3 & 0 \\ 0 & 4 \end{vmatrix}$$

$$= \hat{i}(0+4) - \hat{j}(-3-0) + \hat{k}(12-0)$$

$$= \langle 4, 3, 12 \rangle$$

$$A (x_1, y_1, z_1)$$

$$B (x_2, y_2, z_2)$$

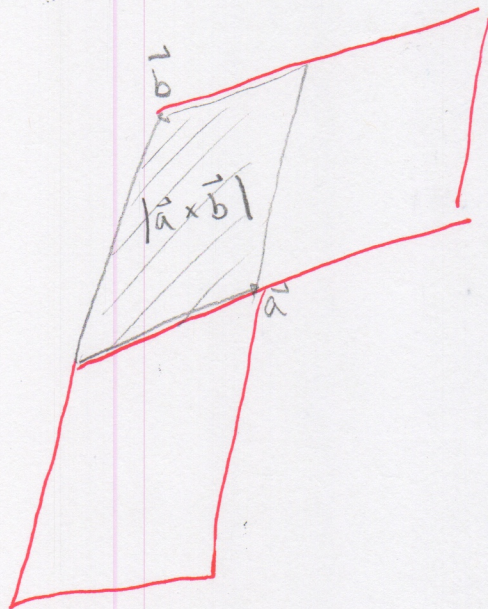
$$\vec{AB} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

PROPERTIES OF CROSS PRODUCT

Let $\vec{a}, \vec{b}, \vec{c}$ be vectors. Let s be a scalar.

$$(1) \vec{a} \times \vec{b} = -\vec{b} \times \vec{a} \quad (\text{ANTI-COMMUTATIVITY})$$

$$(2) (s\vec{a}) \times \vec{b} = \vec{a} \times (s\vec{b}) = s(\vec{a} \times \vec{b})$$



$$(3) \vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c}) \quad \text{LEFT-DISTR. RULE.}$$

$$(4) (\vec{a} + \vec{b}) \times \vec{c} = (\vec{a} \times \vec{c}) + (\vec{b} \times \vec{c}) \quad \text{RIGHT-DISTR. RULE.}$$

(5) "SCALAR TRIPLE PRODUCT"

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{c} \cdot (\vec{a} \times \vec{b}) = \vec{b} \cdot (\vec{c} \times \vec{a})$$

GEOMETRIC INTERPRETATION COMING UP.

(6) "VECTOR TRIPLE PRODUCT"

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{b} \cdot \vec{c})\vec{a} - (\vec{a} \cdot \vec{b})\vec{c}$$

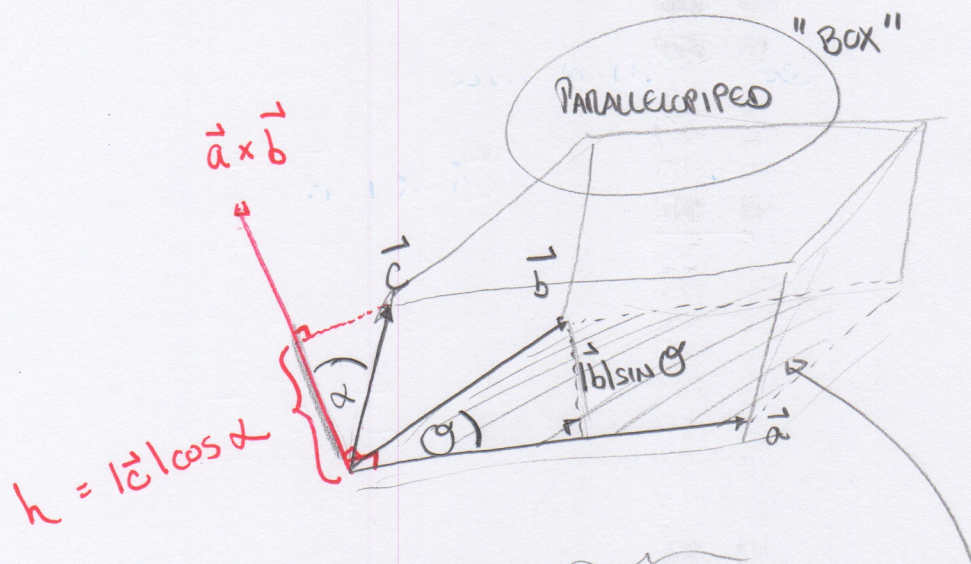
$$\begin{aligned} \vec{a} \times (\vec{b} \times \vec{a}) &= (\vec{b} \cdot \vec{a})\vec{a} \\ &\quad - (\vec{a} \cdot \vec{b})\vec{a} \\ &= 0 \end{aligned}$$

USE PROPERTY 6 TO COMPUTE

$$\vec{a} \times (\vec{b} \times \vec{a}) = ?$$

SCALAR TRIPLE PRODUCT

$$(\vec{a} \times \vec{b}) \cdot \vec{c}$$



$$\begin{aligned} & |(\vec{a} \times \vec{b}) \cdot \vec{c}| \\ &= |\vec{a} \times \vec{b}| |\vec{c}| \cos \alpha \\ &= \text{Area of Base} \times h \end{aligned}$$

AREA OF
BASE $\times$ HEIGHT

$$|\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin \theta = \text{AREA OF PARALLELOGRAM DETERMINED BY } \vec{a} \text{ \& } \vec{b}$$

3 vectors $\vec{a}, \vec{b}, \vec{c}$ DETERMINING A PARALLELEPIPED,
AND VOLUME OF THIS SHAPE IS

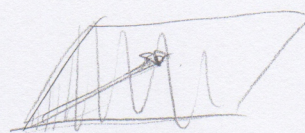
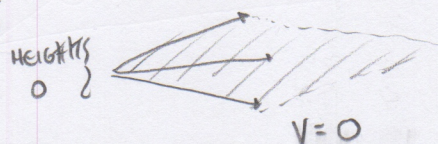
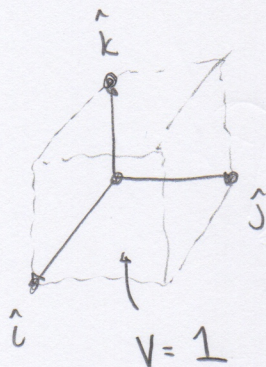
$$V = |(\vec{a} \times \vec{b}) \cdot \vec{c}|$$

RECALL: $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$

$$|\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin \theta$$

ex. SHOW THAT $\langle -2, 1, -2 \rangle$, $\langle 7, -1, 8 \rangle$, & $\langle 5, 0, 6 \rangle$
 ARE COPLANAR

ALL LIE IN ONE PLANE.



3 vectors coplanar $\Leftrightarrow$ SCALAR TRIPLE
 Product is 0

$$\text{SCALAR TRIPLE PRODUCT } (\vec{a} \times \vec{b}) \cdot \vec{c} = \left(\langle -2, 1, -2 \rangle \times \langle 7, -1, 8 \rangle \right) \cdot \langle 5, 0, 6 \rangle$$

$$\begin{pmatrix} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 1 & -2 \\ 7 & -1 & 8 \end{vmatrix} \\ = \hat{i}(8-2) - \hat{j}(-16+14) + \hat{k}(2-7) \\ = \langle 6, 2, -5 \rangle \end{pmatrix}$$

$$= \langle 6, 2, -5 \rangle \cdot \langle 5, 0, 6 \rangle = (6)(5) + (2)(0) + (-5)(6)$$

$$= 30 + 0 - 30$$

$$= 0 \quad \checkmark$$

