

§10.3 THE DOT PRODUCT

2 WAYS TO MULT. VECTORS. ONE WAY: DOT PRODUCT.

Def: GIVEN VECTORS $\vec{a} = \langle x_1, y_1, z_1 \rangle$ & $\vec{b} = \langle x_2, y_2, z_2 \rangle$

$$"\vec{a} \text{ DOT } \vec{b}" = \vec{a} \cdot \vec{b} := \underbrace{x_1 x_2 + y_1 y_2 + z_1 z_2}_{\text{SCALAR!}}$$

ALSO 2D
DEF

DOT PRODUCT = SCALAR PRODUCT = INNER PRODUCT

e.g. $\langle 3, -2 \rangle \cdot \langle 5, 4 \rangle = 3 \cdot 5 + (-2) \cdot 4 = 7.$

e.g. $(\hat{i} + 2\hat{j} + 3\hat{k}) \cdot (-7\hat{i} - \hat{j} + \hat{k}) = -6$

PROPERTIES OF DOT PRODUCT

Let $\vec{a}, \vec{b}, \vec{c}$ be vectors. Let c be scalar.

(1) $\vec{a} \cdot \vec{a} = |\vec{a}|^2$

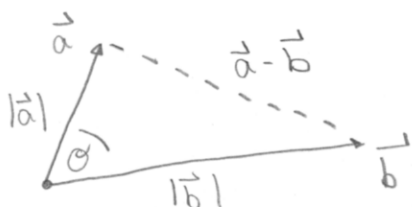
(2) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

(3) $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

(4) $(c\vec{a}) \cdot \vec{b} = c(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (c\vec{b})$

(5) $\vec{0} \cdot \vec{a} = 0$

Geometric Def: of Dot Product: $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$



LAW OF COSINES: $|a-b|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta$

↓

$$(\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b}) = \vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{b} - 2|\vec{a}||\vec{b}|\cos\theta$$

$$\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta \quad \square$$

e.g. FIND THE ANGLE BETWEEN $\langle 1, 0 \rangle$ AND $\langle 1, 1 \rangle$

$\langle 1, 0, 0 \rangle$ AND $\langle 1, 1, 1 \rangle$

CON. IF θ IS ANGLE BETWEEN $\vec{a}$ & $\vec{b}$, THEN

$$\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|}$$

e.g. FIND ACUTE ANGLE OF INTERSECTION FOR $y = 3x + 1$, $y = -x + 3$.

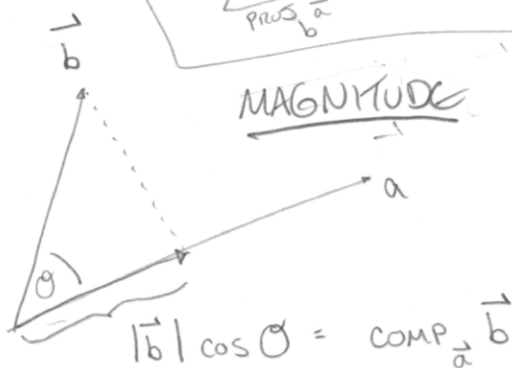
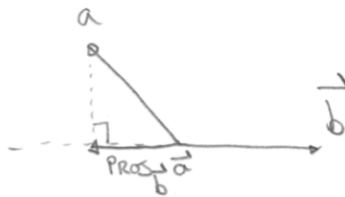
($\perp$, $\parallel$?)



TWO VECTORS ARE ORTHOGONAL IFF THEIR DOT PRODUCT IS ZERO
(PERP.) (i.e. $\theta = \frac{\pi}{2}$)

TWO VECTORS ARE PARALLEL IFF THEIR DOT PRODUCT IS $\pm |\vec{a}| |\vec{b}|$
(i.e. $\theta = 0$ or π)

PROJECTIONS



MAGNITUDE

DIR

$$\frac{\vec{a}}{|\vec{a}|}$$

← UNIT VECTOR.

COMPONENT OF $\vec{b}$ ALONG $\vec{a}$
SCALAR PROJECTION

SCALAR PROJECTION OF

$\vec{b}$ ONTO $\vec{a}$

$$\text{COMP}_{\vec{a}} \vec{b} = |\vec{b}| \cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$$

VECTOR PROJECTION OF

$\vec{b}$ ONTO $\vec{a}$

$$\text{PROJ}_{\vec{a}} \vec{b} = |\vec{b}| \cos \theta \cdot \frac{\vec{a}}{|\vec{a}|}$$

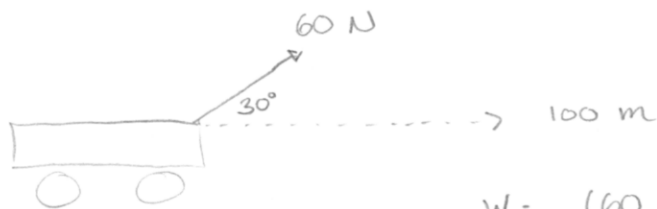
$$= \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} \right) \frac{\vec{a}}{|\vec{a}|} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \vec{a}$$

e.g. FIND SCALAR & VECTOR PROJECTIONS OF $\langle 3, -4, 2 \rangle$ ONTO $\langle -1, 4, 8 \rangle$.

$W = Fd$ WHEN FORCE IS IN THE DIRECTION OF DISPLACEMENT,

$$\text{OTHERWISE, } W = \underbrace{(|\vec{F}| \cos \theta)}_{\text{COMPONENT OF } \vec{F} \text{ ALONG } \vec{D}} |\vec{D}| = \vec{F} \cdot \vec{D}$$

e.g.

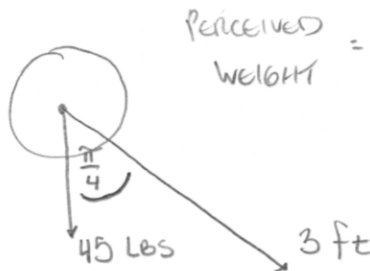


$$W = (60 \text{ N})(100 \text{ m}) \cos \frac{\pi}{6} = 3000\sqrt{3} \text{ J}$$

e.g.

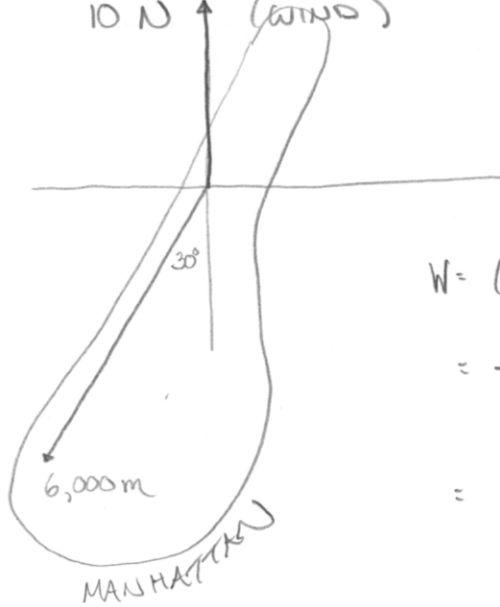


$$W = (45 \text{ lbs})(3 \text{ ft}) \cos \frac{\pi}{4} = \frac{135\sqrt{2}}{2} \text{ J}$$



$$\text{PERCEIVED WEIGHT} = \frac{45}{\sqrt{2}} \approx \underline{\underline{31.8 \text{ LBS.}}}$$

e.g.



BIKE RIDERS
TAKE NOTE!

$$W = (10 \text{ N})(6000 \text{ m}) \cos 150^\circ$$

$$= -(6000) \cos 30^\circ$$

$$= -3000\sqrt{3} \text{ Joules}$$

§12.3 THE DOT PRODUCT (SCALAR PRODUCT)

2 WAYS TO MULTIPLY VECTORS $\vec{v}, \vec{u}$

(1) $\vec{v} \cdot \vec{u}$ " $\vec{v}$ dot $\vec{u}$ " $\rightarrow$ SCALAR

(2) $\vec{v} \times \vec{u}$ " $\vec{v}$ cross $\vec{u}$ " $\rightarrow$ VECTOR

Def: (ANALYTIC) GIVEN 2 VECTORS $\vec{a} = \langle x_1, y_1, z_1 \rangle$
 $\vec{b} = \langle x_2, y_2, z_2 \rangle$

$$\vec{a} \cdot \vec{b} = x_1 x_2 + y_1 y_2 + z_1 z_2 \quad (\text{SINGLE \#, SCALAR})$$

e.g. $\vec{a} = \langle 5, -7 \rangle, \vec{b} = \langle 3, -1 \rangle$

$$\vec{a} \cdot \vec{b} = \langle 5, -7 \rangle \cdot \langle 3, -1 \rangle = \underbrace{(5)(3) + (-7)(-1)} = 15 + 7 = \boxed{22}$$

PROPERTIES OF THE DOT PRODUCT: LET $\vec{a}, \vec{b}, \vec{c}$ BE VECTORS; s BE SCALAR

(1) $\vec{a} \cdot \vec{a} = |\vec{a}|^2$

$x_1 x_1 + y_1 y_1 + z_1 z_1 = \sqrt{x_1^2 + y_1^2 + z_1^2}^2 = |\vec{a}|^2 \quad \checkmark$

(2) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

(3) $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

(4) $(s\vec{a}) \cdot \vec{b} = \vec{a} \cdot (s\vec{b}) = s(\vec{a} \cdot \vec{b})$

(5) $\vec{0} \cdot \vec{a} = \vec{0}$

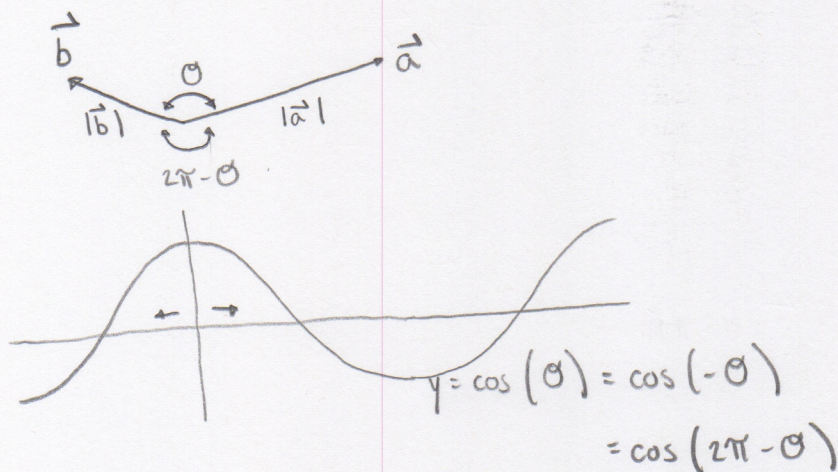
$\langle 0, 0, 0 \rangle \cdot \langle x_1, y_1, z_1 \rangle$

$= 0x_1 + 0y_1 + 0z_1$

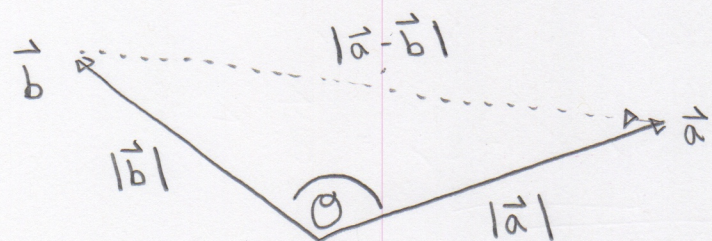
$= 0 \quad \checkmark$

GEOMETRIC DEFINITION OF DOT PRODUCT

$$\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta, \text{ where } \theta \text{ is angle between } \vec{a} \text{ \& } \vec{b}$$



LAW OF COSINES:



$$|\vec{a} - \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta$$

$$(\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b}) =$$

$$(\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b}) = \vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{b} - 2|\vec{a}||\vec{b}|\cos\theta$$

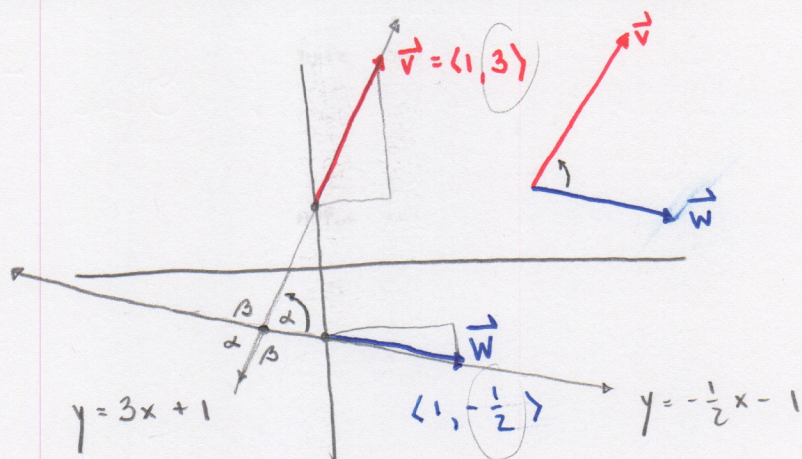
$$\vec{a} \cdot \vec{a} - 2\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b} = \vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{b} - 2|\vec{a}||\vec{b}|\cos\theta$$

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

e.g. FIND THE ACUTE ANGLE OF INTERSECTION BETWEEN THE LINES

$$y = 3x + 1$$

$$y = -\frac{1}{2}x - 1$$



IN GENERAL, THE VECTOR $\langle 1, m \rangle$ IS $\parallel$ TO LINE $y = mx + b$

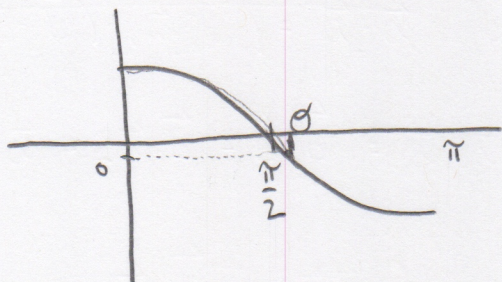
$$\vec{v} \cdot \vec{w} = |\vec{v}| |\vec{w}| \cos \theta$$

$$\langle 1, 3 \rangle \cdot \langle 1, -\frac{1}{2} \rangle = \sqrt{1^2 + 3^2} \sqrt{1^2 + (-\frac{1}{2})^2} \cos \theta$$

$$1 - \frac{3}{2} = \sqrt{10} \sqrt{\frac{5}{4}} \cos \theta$$

$$-\frac{1}{2} = \frac{\sqrt{50}}{2} \cos \theta$$

$$-\frac{1}{\sqrt{50}} = \cos \theta \rightarrow \theta = \cos^{-1}\left(-\frac{1}{\sqrt{50}}\right) = \beta \text{ OBTUSE}$$



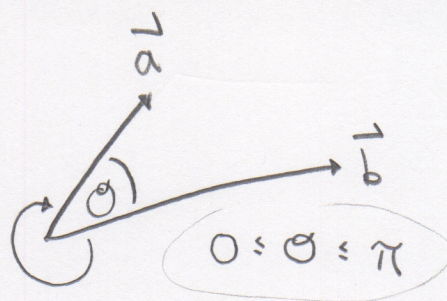
$$\alpha = \pi - \beta$$

$$\alpha = \pi - \cos^{-1}\left(-\frac{1}{\sqrt{50}}\right)$$

§ 12.3 Dot Product

$$\text{Let } \vec{a} = \langle x_1, y_1, z_1 \rangle$$

$$\vec{b} = \langle x_2, y_2, z_2 \rangle$$



$$\vec{a} \cdot \vec{b} = x_1 x_2 + y_1 y_2 + z_1 z_2 = |\vec{a}| |\vec{b}| \cos \theta$$

0 WHEN $\theta = \frac{\pi}{2}$

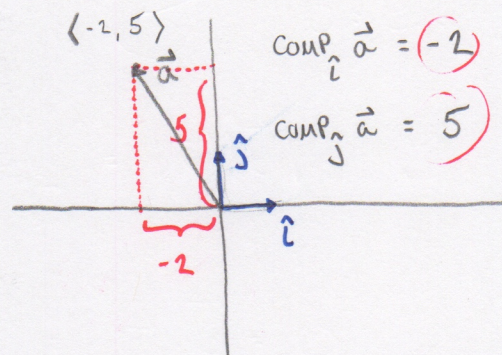
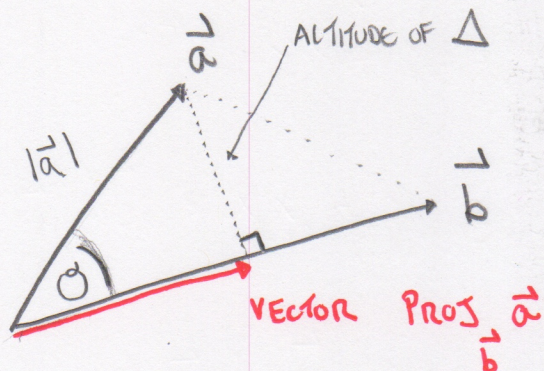
Note: $\vec{a} \perp \vec{b}$ IF AND ONLY IF $\vec{a} \cdot \vec{b} = 0$

$$\theta = \frac{\pi}{2} \rightarrow \cos \theta = 0 \rightarrow \vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| 0 = 0$$

Def: $\vec{a}$ & $\vec{b}$ ARE ORTHOGONAL WHEN
 $\vec{a} \cdot \vec{b} = 0.$

COMPONENTS & PROJECTIONS

How much of $\vec{a}$ points in the direction of $\vec{b}$?



$$\text{COMP}_{\vec{b}} \vec{a} = |\vec{a}| \cos \theta = \frac{|\vec{a}| |\vec{b}| \cos \theta}{|\vec{b}|}$$

$$\text{COMP}_{\vec{b}} \vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$$

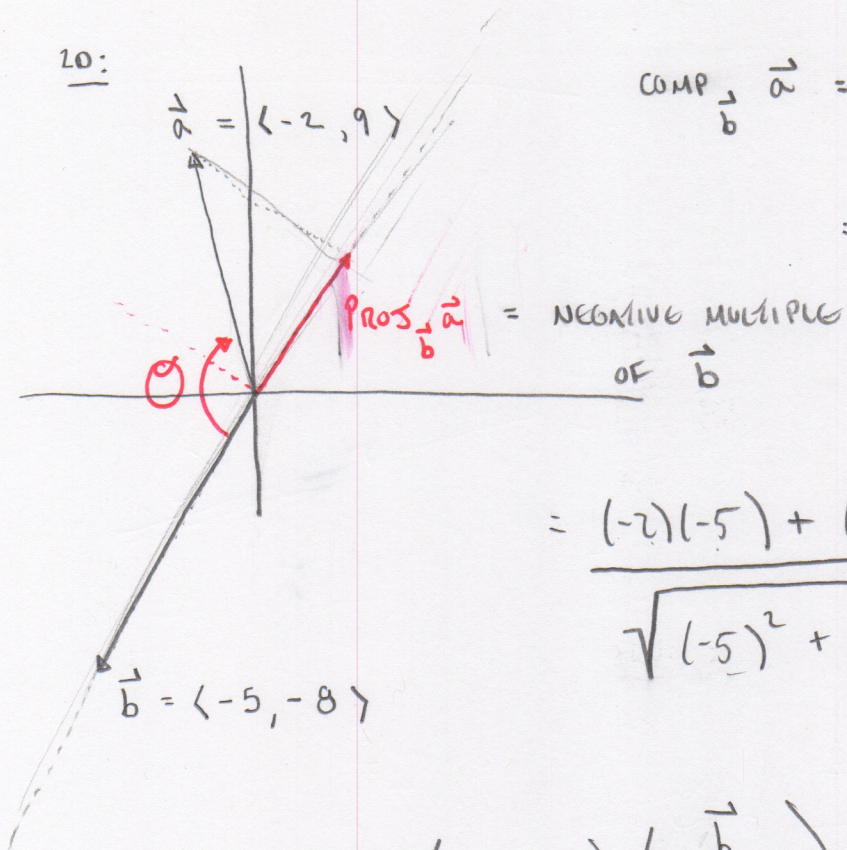
$$\text{PROJ}_{\vec{b}} \vec{a} = \left(\text{LENGTH } \text{COMP}_{\vec{b}} \vec{a} \right) \underbrace{\frac{\vec{b}}{|\vec{b}|}}_{\text{DIRECTION}}$$

UNIT VECTOR IN THE DIRECTION OF $\vec{b}$.

$$\text{PROJ}_{\vec{b}} \vec{a} = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} \right) \frac{\vec{b}}{|\vec{b}|} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2} \vec{b}$$

$$\text{PROJ}_{\vec{b}} \vec{a} = \frac{\vec{a} \cdot \vec{b}}{\vec{b} \cdot \vec{b}} \vec{b}$$

10:



$$\text{comp}_{\vec{b}} \vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$$

$$= \frac{\langle -2, 9 \rangle \cdot \langle -5, -8 \rangle}{|\langle -5, -8 \rangle|}$$

$$= \frac{(-2)(-5) + (9)(-8)}{\sqrt{(-5)^2 + (-8)^2}}$$

$$= \frac{-62}{\sqrt{89}}$$

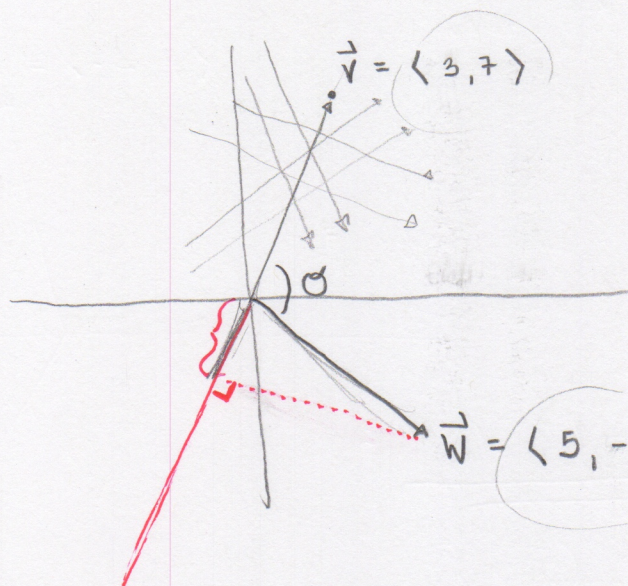
$$\text{proj}_{\vec{b}} \vec{a} = \left(\text{comp}_{\vec{b}} \vec{a} \right) \left(\frac{\vec{b}}{|\vec{b}|} \right)$$

unit vector in direction as $\vec{b}$

$$\text{proj}_{\vec{b}} \vec{a} = \left(\frac{-62}{\sqrt{89}} \right) \left(\frac{\langle -5, -8 \rangle}{\sqrt{89}} \right) = \frac{-62}{89} \langle -5, -8 \rangle$$

$$|\text{proj}_{\vec{b}} \vec{a}| = |\text{comp}_{\vec{b}} \vec{a}| = \begin{cases} \text{comp}_{\vec{b}} \vec{a} & \text{if } 0 \leq \theta \leq \frac{\pi}{2} \\ -\text{comp}_{\vec{b}} \vec{a} & \text{if } \frac{\pi}{2} \leq \theta \leq \pi \end{cases}$$

ex.



COMP _{$\vec{v}$} $\vec{w}$

$$|\vec{w}| = 7.6 \text{ MPH}$$

$$\text{COMP}_{\vec{v}} \vec{w} = \frac{\vec{w} \cdot \vec{v}}{|\vec{v}|} = \frac{\langle 5, -4 \rangle \cdot \langle 3, 7 \rangle}{\sqrt{3^2 + 7^2}} = \frac{15 - 28}{\sqrt{58}} = \frac{-13}{\sqrt{58}}$$

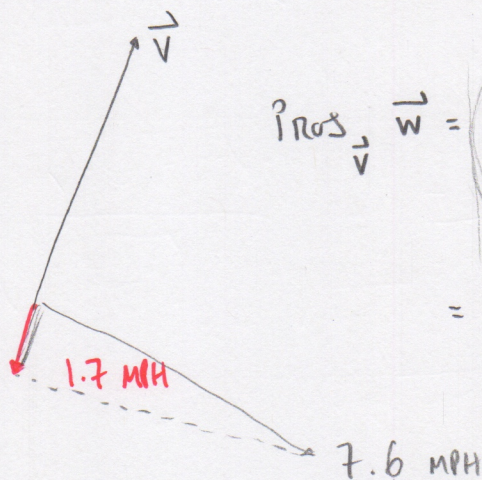
-1.7 MPH

UNIT VECTOR

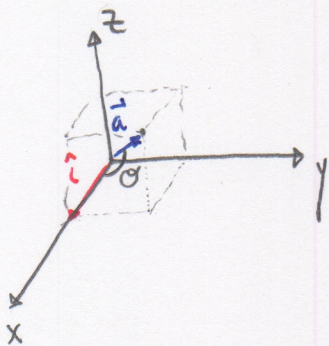
$$\text{PROJ}_{\vec{v}} \vec{w} = \frac{-13}{\sqrt{58}} \left(\frac{\vec{v}}{|\vec{v}|} \right)$$

$$= \frac{-13}{\sqrt{58}} \cdot \frac{1}{\sqrt{58}} \langle 3, 7 \rangle$$

$$= \frac{-13}{58} \langle 3, 7 \rangle$$



e.g. FIND THE ANGLE BETWEEN $\hat{l}$ & $\hat{a}$
 $\langle 1, 0, 0 \rangle$ & $\langle 1, 1, 1 \rangle$



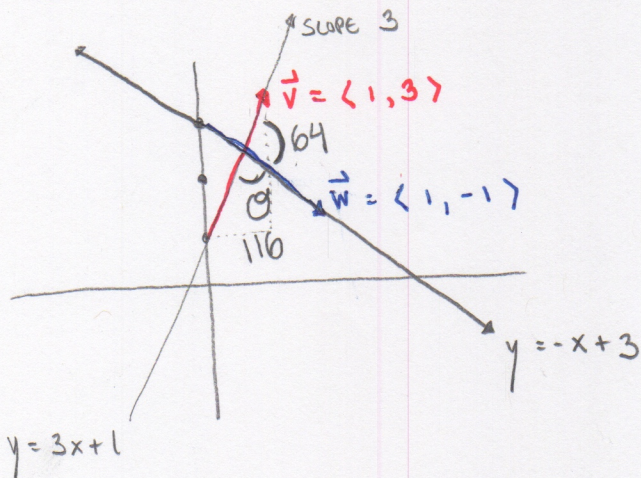
$$\vec{a} \cdot \hat{l} = |\vec{a}| |\hat{l}| \cos \theta$$

$$\langle 1, 1, 1 \rangle \cdot \langle 1, 0, 0 \rangle = \sqrt{1^2 + 1^2 + 1^2} \cos \theta$$

$$1 + 0 + 0 = \sqrt{3} \cos \theta$$

$$\frac{1}{\sqrt{3}} = \cos \theta \rightarrow \theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

e.g. FIND THE ACUTE ANGLE BETWEEN LINES $y = 3x + 1$ ✓
 $y = -x + 3$



GIVEN A LINE WITH SLOPE m ,
 THE VECTOR $\langle 1, m \rangle$ IS
 PARALLEL TO THE LINE.

(SO IS $\langle -1, -m \rangle$)

$$\vec{v} \cdot \vec{w} = |\vec{v}| |\vec{w}| \cos \theta$$

$$\langle 1, 3 \rangle \cdot \langle 1, -1 \rangle = \sqrt{1^2 + 3^2} \sqrt{1^2 + (-1)^2} \cos \theta$$

$$1 - 3 = \sqrt{10} \sqrt{2} \cos \theta$$

$$-2 = \sqrt{20} \cos \theta \rightarrow \cos \theta = -\frac{2}{\sqrt{20}}$$

$$\theta = \cos^{-1}\left(\frac{-2}{\sqrt{20}}\right)$$

$$\pi - \cos^{-1}\left(\frac{-2}{\sqrt{20}}\right)$$

116°

