§10.9 CONVERGENCE OF TAYLOR SERIES

RECALL: IF f HAS DERIVATIVES OF ALL ORDERS ON AN OPEN INTERVAL I CONTAINING a, THEN f HAS A TAYLOR SERIES REPRESENTATION ABOUT $x = a$

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k,$$

AND TAYLOR POLYNOMIALS OF ORDER n GENERATED BY f AT $x = a$

$$\underbrace{P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k}_{\text{APPROXIMATE } f(x)}, \quad \text{FOR } n = 0, 1, 2, \ldots$$

QUESTIONS:

1. FOR WHAT VALUES OF x DOES A TAYLOR SERIES REPRESENTATION OF f CONVERGE TO f?

2. HOW ACCURATELY DO TAYLOR POLYNOMIALS GENERATED BY f APPROXIMATE f?

TAYLOR'S FORMULA

IF f HAS DERIVATIVES OF ALL ORDERS IN AN OPEN INTERVAL I CONTAINING a, THEN FOR EACH POSITIVE INTEGER n & FOR EACH x IN I,

$$(*) \quad f(x) = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k}_{\text{TAYLOR POLYNOMIAL OF ORDER } n} + \underbrace{R_n(x)}_{\text{ERROR ("REMAINDER")}}$$

MISSING PIECE TO MAKE EQUALITY

WHERE $R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$

FOR SOME c BETWEEN a & x. $\left(\begin{array}{c} a \le c \le x \\ x \le c \le a \end{array}\right)$

SO $f(x) = P_n(x) + R_n(x)$ $(*)$

POLYNOMIAL APPROX ERROR

THM:

$$\boxed{\begin{array}{l} \text{IF ERROR} \to 0 \text{ AS } n \to \infty \text{ FOR ALL } x \text{ IN } I \\ \text{THEN THE TAYLOR SERIES GENERATED BY } f \text{ AT } x=a \\ \text{CONVERGES TO } f \text{ ON } I. \end{array}}$$

ex. SHOW THAT THE TAYLOR SERIES GENERATED BY $f(x) = \sin x$ AT $x = 0$

CONVERGES TO f FOR ALL REAL NUMBERS x.

(MACLAURIN SERIES)

FIND TAYLOR SERIES : $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

$f^{(0)}(0) = f(0) = 0$

$f^{(1)}(0) = \cos(0) = 1$

$f^{(2)}(0) = -\sin(0) = 0$

$f^{(3)}(0) = -\cos(0) = -1$

$$\rightarrow \underbrace{= \frac{1}{1!}x^1 - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \ldots}$$

NOW, SHOW THAT THIS CONVERGES FOR ALL x.

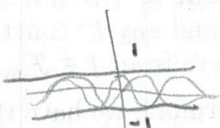
METHOD 1: $I = (-\infty, \infty)$ "FOR ALL x IN I"

SHOW THAT ERROR $\to 0$ AS $n \to \infty$ $\forall x \in I$

ERROR (REMAINDER) $R_n = \dfrac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$

FOR SOME c BETWEEN 0 & x

Note: $\left| f^{(n+1)}(c) \right| = \left| \begin{matrix} \pm \sin(c) \\ \pm \cos(c) \end{matrix} \right| \leq 1$



x FIXED

$$|R_n| \leq \frac{1}{(n+1)!} x^{n+1} \longrightarrow 0 \qquad \underbrace{\underbrace{\underbrace{\frac{x}{1}}_{n=0} \cdot \frac{x}{2}}_{n=1} \cdot \frac{x}{3} \cdot \frac{x}{4}}_{n=3} \cdots \frac{x}{n}$$

$$\lim_{n\to\infty} |R_n| = \boxed{\lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} = 0 \text{ FOR ALL R.N. } x \quad \forall x \in \mathbb{R}}$$

x IS SOME #

$-\infty < x < \infty$

PART 6
THM 5

METHOD 2: $x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

FIND RADIUS OF CONVERGENCE.

$(R = \infty)$

RATIO TEST: $$\rho(x) = \lim_{n\to\infty} \left| \frac{x^{\overset{2n+2+1}{2(n+1)+1}}}{(\underset{2n+2+1}{2(n+1)+1})!} \cdot \frac{(2n+1)!}{x^{2n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\cancel{x^{2n+1}}\, x^2}{(2n+3)(2n+2)\cancel{(2n+1)!}} \cdot \frac{\cancel{(2n+1)!}}{\cancel{x^{2n+1}}} \right|$$

$$\rho(x) = \lim_{n\to\infty} \left| (\dots) \right|$$

$$\underbrace{= \left| \frac{2x-1}{3} \right| < 1}$$

$$\underbrace{x^2 \cdot \lim_{n\to\infty} \left| \frac{1}{(2n+3)(2n+2)} \right|}_{0} = \underline{\underline{0 < 1}}$$

FOR ALL X

$$-1 < \frac{2x-1}{3} < 1$$

$$-\frac{1}{3} < 2x-1 < \frac{1}{3}$$

$\rho(x) = 0$ FOR ALL X

$\boxed{\sum \text{CONV. WHEN}}\ \rho(x) < 1$

i.e. ALWAYS

TAYLOR SERIES : $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sin(x)$	0
1	$\cos(x)$	1
2	$-\sin(x)$	0
3	$-\cos(x)$	-1

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = \sin x$$

2 METHODS (1) SHOW $R_n(x) \to 0$ FOR ALL X AS $n \to \infty$

(2) SHOW THAT RADIUS OF CONVERGENCE IS ∞.

(1) $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$ FOR SOME C BETWEEN 0 & X

$\boxed{f(x) = e^{2x}}$

$f'(x) = 2e^{2x}$

$\vdots$

$f^{(n+1)}(x) = 2^{n+1} e^{2x}$

Note: $\left| f^{(n+1)}(c) \right| = \left| \begin{matrix} \pm \sin(c) \\ \pm \cos(c) \end{matrix} \right| \leq 1$

$\therefore\ 0 \leq \lim_{n\to\infty} |R_n(x)| \leq \lim_{n\to\infty} \frac{2^{n+1} e^{2c}}{(n+1)!} x = 0$ FOR ALL X

$$\boxed{\lim_{n\to\infty} \frac{x^n}{n!} = 0 \quad \text{FOR ALL } X}$$

$$\lim_{n\to\infty} \frac{x^n}{n!} = \underbrace{\underbrace{\underbrace{\frac{x}{1}}_{n=1} \cdot \frac{x}{2}}_{n=2} \cdot \frac{x}{3}}_{n=3} \cdots \frac{x}{1000000} \cdots$$

(2) SHOW RADIUS OF CONVERGENCE IS ∞.

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

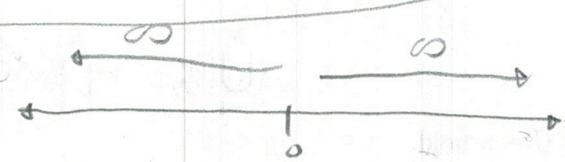
RATIO TEST: $$\rho(x) = \lim_{n\to\infty} \left| \frac{x^{\overset{2n+2+1}{2(n+1)+1}}}{\underset{2n+2+1}{[2(n+1)+1]!}} \cdot \frac{(2n+1)!}{x^{2n+1}} \right|$$

$$= \lim_{\substack{n\to\infty \\ x \text{ FIXED}}} \left| \frac{\cancel{x^{2n+1}}\, x^2}{(2n+3)(2n+2)\cancel{(2n+1)!}} \cdot \frac{\cancel{(2n+1)!}}{\cancel{x^{2n+1}}} \right|$$

$$= x^2 \underbrace{\lim_{n\to\infty} \frac{1}{(2n+3)(2n+2)}}_{0} = 0 \quad \text{NO MATTER WHAT } x \text{ IS.}$$

So $\rho(x) < 1$ FOR ALL x

So $\sum$ CONVERGES FOR ALL x.



RADIUS OF CONVERGENCE $= \infty$.

ex. SHOW THAT THE TAYLOR SERIES GENERATED BY $f(x)=e^x$ AT $\underline{\underline{x=0}}$ CONVERGES TO f FOR ALL REAL NUMBERS x.

FIND TAYLOR SERIES: $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

$f^{(n)}(0) = e^0 = 1$ $\quad = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \dots$

$f^{(n)}(x) = e^x$ $\quad \displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} x^n$

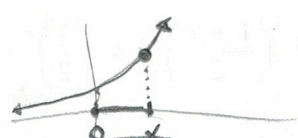
FOR ALL n

$\sum$ CONVERGES TO f AT x IF $\displaystyle\lim_{n\to\infty} R_n(x) = 0$

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$$ WHERE c IS BETWEEN 0 & x

$\boxed{0 \le c \le x}$ (1)

$x \le c \le 0$ (2)



NOTE: $\boxed{f^{(n+1)}(x) = e^x}$ IS INCREASING.

SO ON ANY CLOSED INTERVAL, IT TAKES ITS LARGEST VALUE AT THE RIGHT ENDPOINT

(1) $\displaystyle R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} \le \frac{e^x}{(n+1)!} x^{n+1} \quad \forall n$

$$0 \le \boxed{\lim_{n\to\infty} R_n(x)} \le e^x \cdot \underbrace{\lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!}}_{0} = 0$$

$= 0$ FOR ALL x.

SINCE $R_n(x) \to 0$ AS $n \to \infty$ FOR ALL x, $\sum$ CONVERGES TO f FOR ALL x.

TAYLOR SERIES: $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \dots$

$$f^{(n)}(x) = e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \quad \text{FOR ALL } x!$$

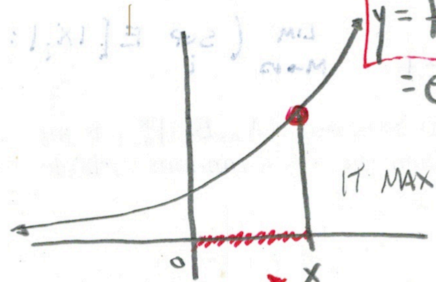
$$f^{(n)}(0) = e^0 = 1$$

SHOW $\lim_{n\to\infty} R_n(x) = 0$ FOR ALL x.

$R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$ WHERE IS SOME # BETWEEN 0 & x.

$0 \le x$ $\qquad\qquad\qquad$ $x \le 0$

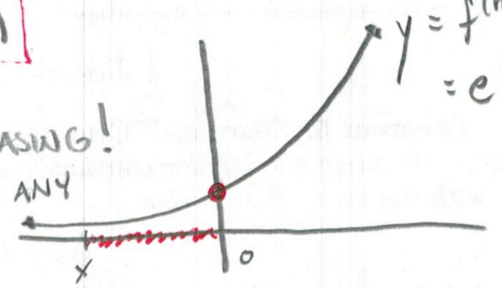
$\boxed{y = f^{(n+1)}(x)}$ $= e^x$



INCREASING!

IT MAXIMUM ON ANY

$y = f^{(n+1)}(x) = e^x$



IF $0 \le c \le x$

MAX ON ANY CLOSED INTERVAL HAPPENS AT THE RIGHT ENDPOINT

$x \le c \le 0$

$$f^{(n+1)}(c) = e^c \le \underline{e^x} \qquad\qquad f^{(n+1)}(c) \le \underline{e^0} = 1$$

$$0 \le \lim_{n\to\infty} \left| \frac{e^c}{(n+1)!} x^{n+1} \right| \le \begin{cases} \lim_{n\to\infty} \dfrac{e^x}{(n+1)!} x^{n+1} & \text{IF } 0 \le x \\ \lim_{n\to\infty} \dfrac{1}{(n+1)!} x^{n+1} & \text{IF } x \le 0 \end{cases}$$

$$0 \le \lim_{n\to\infty} \left| R_n(x) \right| \le \begin{cases} \underbrace{e^x \lim_{n\to\infty} \dfrac{x^{n+1}}{(n+1)!}}_{0} = 0 & \text{IF } 0 \le x \\ \lim_{n\to\infty} \dfrac{x^{n+1}}{(n+1)!} = 0 & \text{IF } x \le 0 \end{cases}$$

SO $\lim_{n\to\infty} R_n(x) = 0$ FOR ALL x

THEREFORE TAYLOR SERIES CONVERGES TO $f(x) = e^x$

FOR ALL x.

SET $x = 1$

$$e = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \dots$$

$$\boxed{e = \sum_{n=0}^{\infty} \frac{1}{n!}}$$

RECALL: $$X^3 \sin(x) = \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \right) X^3$$

$$= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \ldots$$

$$x^3 \sin(x) = x^3 \left[x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \ldots \right]$$

$$= x^4 - \frac{1}{3!}x^6 + \frac{1}{5!}x^8 - \frac{1}{7!}x^{10} + \ldots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+4}$$

ex. FIND THE TAYLOR SERIES GENERATED BY $f(x) = x^3 \sin x$ AT $x = 0$

(AND SHOW IT CONVERGES TO f FOR ALL REAL NUMBERS x.)

RECALL: TAYLOR SERIES FOR $\sin x$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots \quad (\text{FOR ALL } x)$$

$$x^{M} \sin x = x^{M}\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots\right)$$

$$= x^2 - \frac{1}{3!}x^4 + \frac{1}{5!}x^6 - \frac{1}{7!}x^8$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+2}$$