§ 10.7 Power Series (Omit multiplication of series)

N^{TH} degree polynomial:

$$P(x) = \sum_{n=0}^{N} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots + c_N x^N$$

The c_n are the coefficients.

c_0 is the constant term.

∞ degree polynomial ??

<u>Def:</u> A <u>power series about $x = 0$</u> is a series of the form

$$f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$$

A <u>power series about $x = a$</u> is a series of the form

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \ldots$$

We call a the <u>center</u> & the constants $c_0, c_1, c_2, \ldots$ are called the <u>coefficients</u>.

ex. 1 POWER SERIES ABOUT $x = 0$ WITH ALL COEFFICIENTS EQUAL TO 1.

$$f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$$

RECALL GEOMETRIC SERIES :

$$\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \dots$$

$$= \frac{a}{1-r} \quad \text{IF } |r| < 1$$

(DIVERGES OTHERWISE)

SO,
$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$$

CONVERGES TO $\dfrac{1}{1-x}$ IF $|x| < 1$.

IN OTHER WORDS

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots \quad \text{IF } -1 < x < 1$$

IN THIS CASE, WE HAVE 2 WAYS OF EXPRESSING THE SAME FUNCTION.

BUT IN GENERAL, THESE ARE DIFFERENT BECAUSE THEY HAVE DIFFERENT DOMAINS.

ex. 2

FOR WHAT VALUES OF X DOES THE POWER SERIES ABOUT $x=0$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

CONVERGE?

ex. 3

FOR WHAT VALUES OF X DOES THE POWER SERIES ABOUT $x=1$

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{\sqrt{n}}$$

ex. 4

FOR WHAT VALUES OF X DOES THE SERIES

$$\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n}$$

CONVERGE?

CONVERGENCE THM FOR POWER SERIES

IF THE POWER SERIES $\sum_{n=0}^{\infty} c_n x^n$ CONVERGES AT $x = b \neq 0$

THEN IT CONVERGES ABSOLUTELY FOR ALL x SUCH THAT $|x| < |b|$.

IF THE SERIES DIVERGES AT $x = d$,

THEN THE SERIES DIVERGES FOR ALL x SUCH THAT $|x| > |d|$.

PROOF:

PART 1

SUPPOSE $\sum_{n=0}^{\infty} c_n x^n$ CONVERGES AT $x = b$

$\Rightarrow \sum_{n=0}^{\infty} c_n b^n$ CONVERGES

$\Rightarrow \lim_{n\to\infty} c_n b^n = 0$ (DIV. TEST)

$\Rightarrow$ THERE EXISTS AN INTEGER N SUCH THAT FOR $n \geq N$ WE HAVE

$$|c_n b^n| < 1 \quad \Rightarrow \quad |c_n| < \frac{1}{|b|^n}$$

NOW LET $|x| < |b|$, SO $\left|\frac{x}{b}\right| < 1$.

MULTIPLY BOTH SIDES BY $|x|^n$

THEN $$|c_n x^n| < \left|\frac{x}{b}\right|^n$$

$\sum_{n=0}^{\infty} \left|\frac{x}{b}\right|^n$ IS CONVERGENT GEO. SERIES

THEREFORE $\sum_{n=0}^{\infty} c_n x^n$ CONVERGES ABSOLUTELY, $|x| < |b|$.

PART 2

SUPPOSE $\sum_{n=0}^{\infty} c_n x^n$ DIVERGES AT $x = d$.

WE SHOW THAT THE SERIES CANNOT CONVERGE AT ANY x SUCH THAT $|x| > |d|$.

(PROOF BY CONTRADICTION)

ASSUME THE SERIES CONVERGES AT SOME x WITH $|x| > d$.

THEN PART 1 IMPLIES THE SERIES CONVERGES AT $x = d$.

THIS IS A CONTRADICTION. THUS OUR ASSUMPTION MUST BE FALSE. THEREFORE, SERIES DIVERGES AT ALL x WITH $|x| > |d|$.

IF THE POWER SERIES $\sum_{n=0}^{\infty} c_n (x-a)^n$ CONVERGES (DIVERGES) AT $x - a = b \neq 0$, THEN IT CONVERGES (DIVERGES) FOR ALL x SUCH THAT $|x-a| < |b|$ $\left(|x-a| > |b| \right)$.

(REPLACING x WITH $x - a$)

Radius of Convergence

Given a power series $\sum_{n=0}^{\infty} c_n (x-a)^n$,

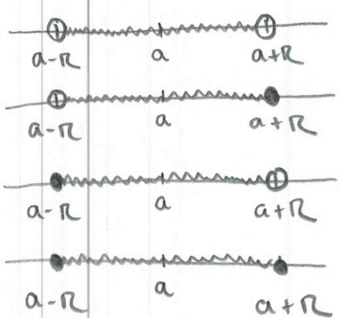
the set of all x such that the series converges

is an "interval" centered at a with radius R.

3 cases: 1. Positive, finite R such that

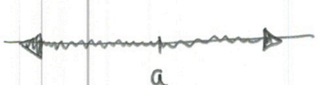
series converges for $|x-a| < R$

diverges for $|x-a| > R$

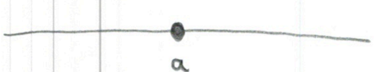


At endpoints $x = a \pm R$ we must check explicitly.

2. If $R = \infty$, then series converges for all x $(-\infty < x < \infty)$.



3. If $R = 0$, then series converges for $x = a$ only.



Def: R is called the <u>radius of convergence</u> of the power series.

HOW TO FIND THE INTERVAL OF CONVERGENCE FOR A POWER SERIES:

1. RATIO / ROOT TEST TO FIND RADIUS OF CONV.
   THIS GIVES LARGEST OPEN INTERVAL ON WHICH
   THE SERIES CONVERGES <u>ABSOLUTELY</u>

$$|x-a| < R \qquad \text{i.e.} \qquad a-R < x < a+R$$

2. IF $0 < R < \infty$, TEST ENDPOINTS
   $x = a \pm R$ SEPARATELY FOR CONVERGENCE/DIVERGENCE
   (D.C.T. , L.C.T. , INT. TEST , ALT. SERIES TEST)

   NOTE: CONVERGENCE AT AN ENDPOINT MAY BE <u>CONDITIONAL</u>.

3. IF $R < \infty$ THEN SERIES DIVERGES FOR $|x-a| > R$.

<u>ex.5</u> FIND RADIUS OF CONV.
AND ALL VALUES x S.T.
POWER SERIES CONVERGES

$$\sum_{n=0}^{\infty} \frac{n(x+3)^n}{5^n}$$

<u>ex.6</u>

FIND RADIUS OF CONV.
AND ALL VALUES x S.T.
POWER SERIES CONV.

$$\sum_{n=1}^{\infty} n^n x^n$$

TERM-BY-TERM DIFFERENTIATION AND INTEGRATION:

SUPPOSE $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ HAS RADIUS OF CONV. R.

$(a-R, a+R)$

THEN f IS DIFFERENTIABLE AND

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1} \quad \text{FOR } x \in (a-R, a+R).$$

AND

$$\int f(x)\,dx = \sum_{n=0}^{\infty} \frac{c_n}{n+1} (x-a)^{n+1} + C$$

FOR $x \in (a-R, a+R)$

ex 7 LET $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$,

WHICH CONVERGES FOR $-1 \leq x \leq 1$ $(R = 1)$.

THEN $f'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \cancel{(2n+1)} x^{2n}}{\cancel{2n+1}} = 1 - x^2 + x^4 - x^6 + \ldots$

GEOMETRIC SERIES

$a = 1$, $r = -x^2$

$$= \frac{1}{1+x^2}$$

$\therefore$ $f(x)$ & $\tan^{-1} x$ HAVE SAME DERIV! $\Rightarrow$ $f(x) = \tan^{-1} x + C$

AND $f(0) = 0$ $\Rightarrow$ $C = 0$ $\therefore$ $f(x) = \tan^{-1} x$.

ex 5 FIND RADIUS OF CONV. AND ALL VALUES x S.T. (SUCH THAT) POWER SERIES CONVERGES

$$\sum_{n=0}^{\infty} \frac{n(x+3)^n}{5^n}$$

ROOT TEST:

$$\rho(x) = \lim_{n\to\infty} \left| \frac{n(x+3)^n}{5^n} \right|^{1/n}$$

$$= \lim_{n\to\infty} \left| \frac{n^{1/n}(x+3)}{5} \right|$$

$\sum$ CONV. ABS. WHEN $\rho < 1$

$$= \left| \boxed{\overset{\infty^0}{\lim_{n\to\infty} n^{1/n}}} \cdot \frac{x+3}{5} \right| = \left| \frac{x+3}{5} \right| < 1$$

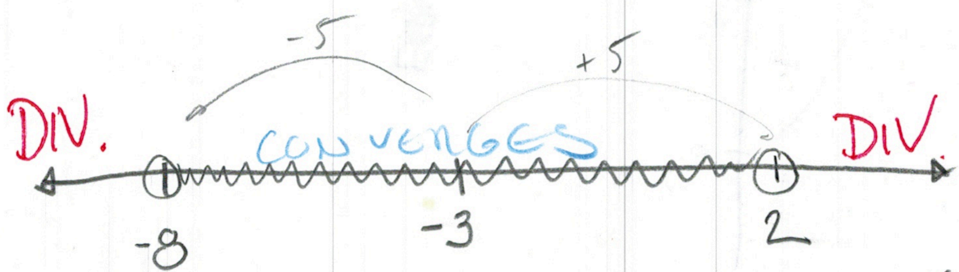
IF ... THEN $\sum$ CONV. ABS.

$$|x+3| < 5$$

$$|x-(-3)| < 5$$

INTERVAL CENTER -3

RADIUS 5



$x=-8$: $\displaystyle\sum_{n=0}^{\infty} \frac{n(-5)^n}{5^n}$

$\displaystyle\sum_{n=0}^{\infty} (-1)^n n$ DIV.

$x=2$: $\displaystyle\sum_{n=0}^{\infty} \frac{n\cancel{(5)^n}}{\cancel{5^n}}$ DIV.

$\sum$ CONVERGES ABS. ON $(-8, 2)$

& DIVERGES ON $(-\infty, -8] \cup [2, \infty)$

ex. 6

FIND RADIUS OF CONV. AND ALL VALUES X S.T. POWER SERIES CONV.

$$\sum_{n=1}^{\infty} n^n x^n$$

ROOT TEST:

$$\rho(x) = \lim_{n\to\infty} \left| n^n x^n \right|^{1/n}$$

$$= \lim_{n\to\infty} |nx|$$

Σ CONV. IF < 1

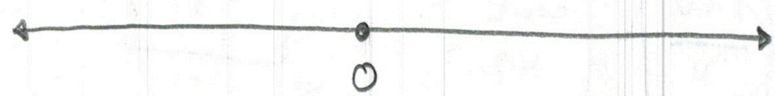
Σ DIV. IF > 1

$$\rho(x) = \underbrace{\lim_{n\to\infty} |nx|}\ < 1$$

NO ENDPOINTS TO CHECK

IF $x = 0$: $\rho(x) = 0$.

IF $x \neq 0$: $\rho(x) = \infty$.



Σ CONVERGES AT

$x = 0$ ONLY

RADIUS OF CONV. $R = 0$.