§10.1 SEQUENCES

DEF: A SEQUENCE IS A LIST OF NUMBERS IN A GIVEN ORDER:

$$a_1, a_2, a_3, \ldots, a_{n-1}, a_n, a_{n+1}, \ldots$$

a_1: 1ST TERM, a_2: 2ND TERM, a_n: nTH TERM

ex. SEQUENCE: $1, 3, 5, 7, 9, \ldots$

$$a_1 = 1, \quad a_2 = 3, \quad a_3 = 5, \quad \ldots, \quad a_n = 2n-1$$

n IS CALLED THE INDEX

INFINITE SEQUENCE (OUR FOCUS) → A SEQUENCE IS A FUNCTION THAT ASSIGNS TO EVERY POSITIVE INTEGER INDEX (n) A TERM (a_n).

NOTATION: $a_n = (-1)^{n+1} \frac{1}{n}$

$$\{a_n\} = \{1, -\tfrac{1}{2}, \tfrac{1}{3}, -\tfrac{1}{4}, \ldots\}$$

$$\{a_n\} = \{(-1)^{n-1} \tfrac{1}{n}\}_{n=1}^{\infty}$$

GRAPHS OF SEQUENCE (DOMAIN $\mathbb{N}$) ex: MATLAB

Def: THE SEQUENCE $\{a_n\}$ CONVERGES TO THE LIMIT L IF

$\forall \varepsilon > 0 \ \exists N$ s.t. $n > N \Rightarrow |a_n - L| < \varepsilon$.

FOR ALL $\varepsilon > 0$, THERE EXISTS A POSITIVE INTEGER N SUCH THAT

WHENEVER $\underbrace{n > N}_{\text{IF } n \text{ IS BIG ENOUGH}}$ WE HAVE $\underbrace{|a_n - L| < \varepsilon}_{a_n \text{ IS CLOSE ENOUGH TO } L}$.

WE WRITE:

$\lim_{n\to\infty} a_n = L$, OR

$a_n \to L$ (AS $n \to \infty$)

IF NO SUCH NUMBER L EXISTS, WE SAY $\{a_n\}$ DIVERGES.

(ex: $\lim_{n\to\infty} \frac{1}{n} = 0$)

- $\pm\infty$
- OSCILLATING BETWEEN 2 OR MORE VALUES

(JUST LIKE LIMITS WE ARE FAMILIAR WITH EXCEPT DOMAIN OF FUNCTION (SEQ) IS $\mathbb{N}$, NOT $\mathbb{R}$.)

THM 1

CALCULATING LIMITS OF SEQUENCES:

LET $\{a_n\}, \{b_n\}$ BE SEQUENCES OF REAL #'S. LET $A, B \in \mathbb{R}$ s.t. $\lim_{n\to\infty} a_n = A$, $\lim_{n\to\infty} b_n = B$.

1. $\lim_{n\to\infty} (a_n \pm b_n) = A \pm B$
2. $\lim_{n\to\infty} (ca_n) = cA$, $\forall c \in \mathbb{R}$.
3. $\lim_{n\to\infty} (a_n b_n) = AB$
4. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$

ex. FIND LIMIT

- $a_n = \frac{1-5n^4}{n^4+8n^3}$
- $a_n = \left(\frac{n+1}{2n}\right)\left(\quad \right)$

THM 2

SANDWICH THM

LET $\{a_n\}, \{b_n\}, \{c_n\}$ BE SEQ. OF REAL #'S.

IF $a_n \le b_n \le c_n$ HOLDS FOR ALL n LARGER THAN SOME INDEX N,

AND IF $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, THEN $\lim_{n\to\infty} b_n = L$ ALSO.

ex. $a_n = \frac{\cos(\pi n)}{n^2}$, $a_n = \frac{(-1)^{n+1}}{2^n}$

THM 3

CONTINUOUS FUNC. THM FOR SEQUENCES

LET $\{a_n\}$ BE A SEQUENCE OF REAL #'s.

IF $a_n \to L$ AND IF f IS A FUNCTION CONTINUOUS AT L

AND DEFINED AT L THEN $f(a_n) \to f(L)$

$$\left(\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) \right)$$

ex. $\lim_{n\to\infty} \sqrt{\dfrac{3n^2+1}{2n^2-1}}$

THM 4

SUPPOSE $f(x)$ IS A FUNCTION DEFINED $\forall x \geq n_0$ AND $\{a_n\}$ IS A SEQUENCE OF REAL NUMBERS S.T. $a_n = f(n)$ FOR $n \geq n_0$.

THEN $\lim_{n\to\infty} a_n = L$ WHENEVER $\lim_{x\to\infty} f(x) = L$.

ex. $a_n = \dfrac{\ln(n)}{n}$ (L'HOSPITAL'S RULE)

ex. $a_n = \left(\dfrac{n+1}{n-1}\right)^n$ $\qquad \ln(a_n) = n \ln\left(\dfrac{n+1}{n-1}\right) = \dfrac{\ln\left(\frac{n+1}{n-1}\right)}{1/n}$

L'Ho: $\cancel{\dfrac{\dfrac{(n-1)-(n+1)}{(n-1)^2}}{-n^{-2}}}$ $\qquad \cancel{\dfrac{-2}{(n-1)^2} \cdot (-n^2)}$

$\cancel{\dfrac{2n^2}{n^2}}$

THM 5 COMMONLY OCCURRING LIMITS (BOOK p. 584)

Def: SEQUENCES CAN BE DEFINED RECURSIVELY:

1. VALUE(S) OF INITIAL TERM(S) GIVEN

2. A RULE CALLED RECURSION FORMULA IS GIVEN FOR CALCULATING ANY LATER TERM FROM TERMS THAT PRECEDE IT.

ex. FIBONACCI, GEOMETRIC,

Def: UPPER BOUND / LOWER BOUND

LEAST UPPER BOUND / GREATEST LOWER BOUND

BOUNDED VS. UNBOUNDED (HAS BOTH)

i) NON DECREASING : $a_n \leq a_{n+1}$

i) NON INCREASING : $a_n \geq a_{n+1}$

i) MONOTONIC = NON-INCR OR NON-DECR

~~THM 6 BOUNDED MONOTONIC SEQ. CONVERGE.~~

EVERY SEQUENCE WITH AN UPPER BOUND HAS A L.U.B.

~~LOWER~~

EVERY SEQUENCE WITH A LOWER BOUND HAS A G.L.B

THM 6: IF A SEQ $\{a_n\}$ IS BOTH BOUNDED AND MONOTONIC, THEN THE SEQUENCE CONVERGES.