Name: *Answer Key* 4/2/2020
Math 20200-PR/RS: Calculus II **Exam 2**

Please print this exam and take take it in a quiet environmnt as you would take an exam in class – without any assistance. You should be able to complete this exam in under 2 hours. When you are done, you should compare your answers to the solutions posted at `johnadamski.com/212s2020.html`.

1. Determine whether the sequence converges or diverges. If the sequence converges, find the limit of the sequence.

$$a_n = \frac{\sin n}{n}$$

$-1 \le \sin n \le 1$ FOR ALL $n \ge 1$

$$\Rightarrow \lim_{n\to\infty} \frac{-1}{n} \le \lim_{n\to\infty} \frac{\sin n}{n} \le \lim_{n\to\infty} \frac{1}{n}$$

"SANDWICH THEOREM"

$$\Rightarrow 0 \le \lim_{n\to\infty} \frac{\sin n}{n} \le 0$$

$$\Rightarrow \lim_{n\to\infty} \frac{\sin n}{n} = \boxed{0 \text{ (CONVERGES)}}$$

2. For each of the following series, determine whether the series converges or diverges. If the series converges, find the sum of the series.

(a) $$\sum_{n=0}^{\infty} e^{-2n} = \underbrace{1 + e^{-2} + e^{-4} + e^{-6} + \dots}_{\text{GEOMETRIC SERIES}: \ a = 1, \ r = e^{-2}}$$

$$\left(a + ar + ar^2 + ar^3 + \dots = \frac{a}{1-r}, \text{ ASSUMING } |r| < 1 \right)$$

SINCE $-1 < r = e^{-2} < 1$,

THE GEOMETRIC SERIES CONVERGES TO $\boxed{\dfrac{1}{1-e^{-2}}}$

(b) $\sum_{n=1}^{\infty} \ln \frac{1}{3^n}$

THE n^{TH} TERM OF THE SERIES IS $a_n = \ln \frac{1}{3^n}$

AND $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln \frac{1}{3^n} = \lim_{n\to\infty} n \ln \frac{1}{3}$

$= \ln \frac{1}{3} \cdot \lim_{n\to\infty} n = -\infty$

(DOES NOT EXIST)

$\boxed{\text{DIVERGES BY DIVERGENCE TEST}}$

3. For each of the following series, determine whether the series converges absolutely, converges conditionally, or diverges. State the name of any tests for convergence that you use, e.g. divergence test, integral test, p-test, direct comparison test, limit comparison test, ratio test, root test, alternating series test.

(a) $\sum_{n=1}^{\infty} \frac{1}{n[1+(\ln n)^2]}$

$f(x) = \frac{1}{x\left[1+(\ln x)^2\right]}$ IS A CONTINUOUS FUNCTION ON $[1, \infty)$

AND $f(n) = a_n = \frac{1}{n\left[1+(\ln n)^2\right]}$ FOR $n = 1, 2, \ldots$

SINCE $\int_1^{\infty} \frac{1}{x\left[1+(\ln x)^2\right]}\,dx = \lim_{t\to\infty} \int_0^t \frac{1}{1+u^2}\,du$ $\left(\begin{array}{l} u = \ln x \\ du = \frac{1}{x}\,dx \end{array}\right)$

$= \lim_{t\to\infty} \tan^{-1} t = \frac{\pi}{2}$ CONVERGES,

THE SERIES ALSO $\boxed{\text{CONVERGES BY THE INTEGRAL TEST.}}$

(b) $\sum_{n=1}^{\infty} \frac{n^5+4}{\sqrt{9n^6-4}}$

SINCE $\frac{n^5+4}{\sqrt{9n^6-4}} \geq \frac{n^5}{\sqrt{9n^6}} = \frac{n^5}{3n^3} = \frac{n^2}{3}$

AND $\sum_{n=1}^{\infty} \frac{n^2}{3}$ DIVERGES (BY DIVERGENCE TEST

OR p-TEST WITH $p=-2$)

THE SERIES ALSO $\boxed{\text{DIVERGES BY THE DIRECT COMPARISON TEST.}}$

(c) $\sum_{n=1}^{\infty} \frac{n+2^n}{n^2 2^n} = \sum_{n=1}^{\infty} \frac{n}{n^2 2^n} + \sum_{n=1}^{\infty} \frac{2^n}{n^2 2^n}$

IF THESE BOTH CONVERGE, THEN THE SERIES CONVERGES.

SINCE $0 \leq \frac{n}{n^2 2^n} \leq \frac{1}{2^n}$ FOR ALL n AND $\sum_{n=1}^{\infty} \frac{1}{2^n}$ CONVERGES

(GEOMETRIC WITH $r=\frac{1}{2}$), $\sum_{n=1}^{\infty} \frac{n}{n^2 2^n}$ ALSO CONVERGES BY DIRECT COMPARISON THEOREM.

$\sum_{n=1}^{\infty} \frac{2^n}{n^2 2^n} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ CONVERGES BY p-TEST WITH $p=2$.

THEREFORE $\sum_{n=1}^{\infty} \frac{n+2^n}{n^2 2^n}$ $\boxed{\text{CONVERGES.}}$

(d) $\sum_{n=1}^{\infty} \frac{(-3)^n}{n^3 2^n}$

THE TERMS DON'T GO TO 0.

(*)

$$\lim_{n\to\infty} \frac{(-3)^n}{n^3 2^n} = 0 \text{ IF AND ONLY IF } \lim_{x\to\infty} \left| \frac{(-3)^n}{n^3 2^n} \right| = 0$$

$$(*) = \lim_{x\to\infty} \frac{(3/2)^x}{x^3} = \lim_{x\to\infty} \frac{(3/2)^x \ln(3/2)}{3x^2} \quad \text{(L'HOPITAL'S RULE)}$$

$$= \lim_{x\to\infty} \frac{(3/2)^x \ln(3/2)^2}{6x} = \lim_{x\to\infty} \frac{(3/2)^x \ln(3/2)^3}{6} = \infty$$

THEREFORE, THE SERIES $\boxed{\text{DIVERGES BY THE DIVERGENCE TEST.}}$

(e) $\sum_{n=1}^{\infty}(-1)^{n+1} \underbrace{\frac{1+n}{n^2}}_{u_n}$

$$\text{SINCE} \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1+n}{n^2} = 0$$

THE SERIES CONVERGES BY THE ALTERNATING SERIES TEST.

HOWEVER, $\sum_{n=1}^{\infty} \frac{1+n}{n^2} \geq \sum_{n=1}^{\infty} \frac{n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n}$ DIVERGES BY p-TEST.

THEREFORE THE SERIES DOES NOT CONVERGE ABSOLUTELY.

THUS THE SERIES $\boxed{\text{CONVERGES CONDITIONALLY.}}$

4. For each of the following power series, find the interval of convergence (including possible endpoints).

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-1)^n}{n3^n}$

ROOT TEST: $\displaystyle \rho(x) = \lim_{n\to\infty} \left| \frac{(x-1)^n}{n\,3^n} \right|^{1/n}$

$$= \lim_{n\to\infty} \frac{|x-1|}{3} \cdot \underbrace{\frac{1}{n^{1/n}}}_{1} < 1$$

$|x-1| < 3 \Rightarrow -3 < x-1 < 3 \Rightarrow -2 < x < 4$ CONVERGES

ENDPOINTS: $\displaystyle\sum_{n=1}^{\infty} \frac{(-2-1)^n}{n\,3^n} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ CONVERGES

$\displaystyle\sum_{n=1}^{\infty} \frac{(4-1)^n}{n\,3^n} = \sum_{n=1}^{\infty} \frac{1}{n}$ DIVERGES.

$\boxed{[-2, 4)}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{nx^n}{4^n(n^2+1)}$

RATIO TEST: $\displaystyle \lim_{n\to\infty} \left| \frac{(n+1)x^{\cancel{n+1}}}{4^{\cancel{n+1}}[(n+1)^2+1]} \cdot \frac{\cancel{4^n}(n^2+1)}{n\cancel{x^n}} \right|$

$$= \lim_{n\to\infty} \underbrace{\frac{n+1}{n}}_{1} \cdot \underbrace{\frac{n^2+1}{(n+1)^2+1}}_{1} \cdot \frac{|x|}{4} < 1 \Rightarrow |x| < 4$$

$-4 < x < 4$

ENDPOINTS: $\displaystyle\sum_{n=1}^{\infty} \frac{(-4)^n n}{4^n (n^2+1)} = \sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1}$ CONVERGES BY ALT. SERIES TEST.

$\displaystyle\sum_{n=1}^{\infty} \frac{\cancel{4^n}}{\cancel{4^n}(n^2+1)} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$ CONVERGES BY p-TEST, $p=2$.

$\boxed{[-4, 4]}$

5. (a) Find the Taylor series centered at $x = 0$ (i.e. Maclaurin series) generated by

$$f(x) = \frac{1}{1+x}.$$

n	$f^{(n)}(x)$	$\frac{f^{(n)}(0)}{n!}$
0	$(1+x)^{-1}$	1
1	$-(1+x)^{-2}$	-1
2	$2(1+x)^{-3}$	1
3	$(-3)(2)(1+x)^{-4}$	-1
$\vdots$		
n	$(-1)^n n!(1+x)^{-n-1}$	$(-1)^n$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$= \boxed{\sum_{n=0}^{\infty} (-1)^n x^n}$$

(b) Use your answer to part (a) to find the taylor series centered at $x = 0$ for

$$F(x) = \ln(1+x).$$

Hint: $F'(x) = f(x)$. That is, F is an antiderivative of f.

$$F(x) = \int f(x)\,dx = \int 1 - x + x^2 - x^3 + \dots \, dx$$

$$= C + x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots$$

SINCE $F(0) = 0$, $C = 0$.

$$F(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots$$

$$\boxed{F(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n}$$