* Answer Key *

Name: Answer Key 2/27/2020

Math 212: Calculus II Exam 1

Please put away all papers and electronic devices. Show enough work that it is clear how you arrived at your answer. Put a box/circle around your final answer to each question. Good luck!

1. Consider the following function of x.

DEFINITION

$$F(x) = \int_1^x \frac{1}{t}\,dt = \ln x$$

Hint: there is a common name for this function.

(a) (6 points) Evaluate the derivative $\frac{d}{dx}F(x)$. $= \frac{d}{dx}[\ln x] = \boxed{\frac{1}{x}}$

(ALT: BY FUND. THM. OF CALC. $\frac{d}{dx}\left[\int_1^x \frac{1}{t}\,dt\right] = \frac{1}{x}$)

(b) (6 points) Evaluate the integral $\int_1^e F(x)\,dx$. $= \int_1^e \ln x\,dx$ INT. BY PARTS

$u = \ln x \quad v = x$

$du = \frac{1}{x}dx \quad dv = dx$

$$= x\ln x\Big|_1^e - \int x \cdot \frac{1}{x}\,dx$$

$$= \Big[x\ln x - x\Big]_1^e = e\underbrace{\ln e}_{1} - e - (1\underbrace{\ln 1}_{0} - 1)$$

$$= e - e - 0 + 1 = \boxed{1}$$

2. (6 points) Use the definitions of $\sinh x$ and $\cosh x$ to simplify the following expression.

$$\ln(\cosh x + \sinh x) + \ln(\cosh x - \sinh x)$$

$$= \ln\left(\frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2}\right) + \ln\left(\frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2}\right)$$

$$= \ln\left(\frac{\cancel{2}e^x}{\cancel{2}}\right) + \ln\left(\frac{\cancel{2}e^{-x}}{\cancel{2}}\right)$$

$$= x - x = \boxed{0}$$

(ALT: $\ln A + \ln B = \ln(AB)$

$\Rightarrow$ EXPRESSION $= \ln(\cosh^2 x - \sinh^2 x)$

$= \ln(1) = \boxed{0}$)

$$\tan^4 x = (\tan^2 x)^2 = (\sec^2 x - 1)^2$$

3. Evaluate the integrals.

(a) (8 points) $\displaystyle\int \sec^3 x \tan^5 x \, dx$ $= \displaystyle\int \sec^2 x \tan^4 x \sec x \tan x \, dx$

$$= \int \sec^2 x \, (\sec^2 x - 1)^2 \sec x \tan x \, dx$$

$u = \sec x$

$du = \sec x \tan x \, dx$

$$\rightsquigarrow \int u^2 (u^2 - 1)^2 \, du = \int u^6 - 2u^4 + u^2 \, du$$

$$= \frac{1}{7}u^7 - \frac{2}{5}u^5 + \frac{1}{3}u^3 + C$$

$$\rightsquigarrow \boxed{\frac{1}{7}\sec^7 x - \frac{2}{5}\sec^5 x + \frac{1}{3}\sec^3 x + C}$$

(b) (8 points) $\displaystyle\int (x^2+1)e^{2x} \, dx$ $= \displaystyle\int u \, dv$

$u = x^2 + 1$ $\quad v = \frac{1}{2}e^{2x}$

$du = 2x \, dx$ $\quad dv = e^{2x} \, dx$

$$= uv - \int v \, du$$

$$= \frac{1}{2}(x^2+1)e^{2x} - \underbrace{\int x e^{2x} \, dx}$$

$u = x$ $\quad v = \frac{1}{2}e^{2x}$

$du = dx$ $\quad dv = e^{2x} \, dx$

$$= \frac{1}{2}(x^2+1)e^{2x} - \left[\frac{1}{2}x e^{2x} - \int \frac{1}{2} e^{2x} \, dx \right]$$

$$= \boxed{\frac{1}{2}(x^2+1)e^{2x} - \frac{1}{2}x e^{2x} + \frac{1}{4}e^{2x} + C}$$

$$= \frac{1}{4}e^{2x}\left(2x^2 - 2x + 3\right) + C$$

4. Evaluate the integrals.

(a) (8 points) $\int e^{-x}\sin(\pi x)\,dx$

$u = \sin \pi x \qquad v = -e^{-x}$

$du = \pi \cos \pi x \, dx \qquad dv = e^{-x}\,dx$

$$\int e^{-x} \sin \pi x \, dx = -e^{-x} \sin \pi x + \pi \int e^{-x} \cos \pi x \, dx$$

$u = \cos \pi x \qquad v = -e^{-x}$

$du = -\pi \sin \pi x \, dx \qquad dv = e^{-x}\,dx$

$$\int e^{-x} \sin \pi x \, dx = -e^{-x} \sin \pi x + \pi \left[-e^{-x} \cos \pi x - \pi \int e^{-x} \sin \pi x \, dx \right]$$

$$(1+\pi^2) \int e^{-x} \sin \pi x \, dx = -e^{-x} \sin \pi x - \pi e^{-x} \cos \pi x$$

$$\int e^{-x} \sin \pi x \, dx = \boxed{\frac{-e^{-x}(\sin \pi x + \pi \cos \pi x)}{1+\pi^2} + C}$$

(b) (8 points) $\int_0^1 \sin^{-1} x \, dx$

$u = \sin^{-1} x \qquad v = x$

$du = \frac{1}{\sqrt{1-x^2}}\,dx \qquad dv = dx$

$$= x \sin^{-1} x \Big|_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx$$

$u = 1 - x^2$

$du = -2x\,dx \rightarrow -\frac{1}{2}du = x\,dx$

$$= x \sin^{-1} x \Big|_0^1 + \frac{1}{2} \int_1^0 \frac{1}{\sqrt{u}}\,du = x \sin^{-1} x \Big|_{x=0}^{x=1} + \sqrt{u}\Big|_{u=1}^{u=0}$$

$$= \left(1 \sin^{-1}(1) - 0\right) + \left(0 - \sqrt{1}\right) = \boxed{\frac{\pi}{2} - 1}$$

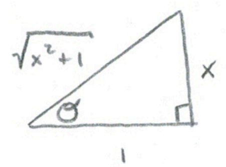
$$\frac{\sec\theta}{\tan^2\theta} = \frac{1}{\cancel{\cos\theta}} \cdot \frac{\cos^{\cancel{2}}\theta}{\sin^2\theta}$$

5. Evaluate the integrals.

(a) (8 points) $\displaystyle\int \frac{1}{x^2\sqrt{x^2+1}}\,dx$

Let $x = \tan\theta$

$dx = \sec^2\theta\, d\theta$



$$\rightsquigarrow \int \frac{\sec^{\cancel{2}}\theta}{\tan^2\theta\ \cancel{\sec\theta}}\,d\theta$$

$\left(\sqrt{x^2+1} = \sec\theta\right)$

$$= \int \frac{\cos\theta}{\sin^2\theta}\,d\theta$$

$u = \sin\theta$

$du = \cos\theta\, d\theta$

$$\rightsquigarrow \int \frac{1}{u^2}\,du = -\frac{1}{u} + C \rightsquigarrow -\csc\theta + C$$

$$\rightsquigarrow \boxed{-\frac{\sqrt{x^2+1}}{x} + C}$$

(b) (8 points) $\displaystyle\int \frac{x+3}{2x^3+8x}\,dx$

$$\frac{x+3}{2x(x^2+4)} = \frac{A}{2x} + \frac{Bx+C}{x^2+4}$$

$$x+3 = A(x^2+4) + (Bx+C)(2x)$$

$$x+3 = (A+2B)x^2 + 2Cx + 4A$$

$\therefore$

$A + 2B = 0$

$2C = 1$

$4A = 3$

$A = \frac{3}{4}$, $B = -\frac{3}{8}$, $C = \frac{1}{2}$

$$= \frac{3}{8}\int \frac{1}{x}\,dx - \frac{3}{8}\int \frac{x}{x^2+4}\,dx + \frac{1}{2}\int \frac{1}{x^2+4}\,dx$$

$$= \boxed{\frac{3}{8}\ln|x| - \frac{3}{16}\ln|x^2+4| + \frac{1}{4}\tan^{-1}\left(\frac{x}{2}\right) + C}$$

6. (6 points) Give the *form* of the partial fraction decomposition for the following rational function. Leave your answer in terms of constant coeficients A, B, C, etc., which you *do not* need to solve for.

$$\frac{3x^5 - 2x + 1}{(x-1)^3(x^2+3)^2}$$

$$\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{Dx+E}{x^2+3} + \frac{Fx+G}{(x^2+3)^2}$$

7. Consider the following integral

$$\int_0^2 \frac{x+1}{\sqrt{4-x^2}}\,dx$$

(a) (4 points) Briefly explain why the integral is improper.

THE INTEGRAND HAS A DISCONTINUITY AT $x = 2$.

(b) (8 points) The integral converges. What does it converge to?

$$\lim_{t \to 2^-} \int_0^t \frac{x+1}{\sqrt{4-x^2}}\,dx = \lim_{t \to 2^-} \left[\int_0^t \frac{x}{\sqrt{4-x^2}}\,dx + \int_0^t \frac{1}{\sqrt{4-x^2}}\,dx \right]$$

$$= \lim_{t \to 2^-} \left[-\sqrt{4-x^2}\,\Big|_0^t + \sin^{-1}\left(\frac{x}{2}\right)\Big|_0^t \right]$$

$$= \lim_{t \to 2^-} \left[\underbrace{\left(-\sqrt{4-t^2} + 2\right)}_{\downarrow\, 0} + \underbrace{\left(\sin^{-1}\left(\frac{t}{2}\right) - 0\right)}_{\frac{\pi}{2}} \right]$$

$$= \boxed{2 + \frac{\pi}{2}}$$

8. Use an appropriate test for convergence to decide whether each of the following improper integrals converges or diverges.

(a) (8 points) $\displaystyle\int_2^\infty \frac{1}{\sqrt{x} - e^{-x}}\,dx$

$\sqrt{x} - e^{-x} < \sqrt{x}$, FOR $x \geq 2$

$\dfrac{1}{\sqrt{x} - e^{-x}} > \dfrac{1}{\sqrt{x}}$, FOR $x \geq 2$

$$\Rightarrow \int_2^\infty \frac{1}{\sqrt{x} - e^{-x}}\,dx > \int_2^\infty \frac{1}{x^{1/2}}\,dx$$

THIS INTEGRAL DIVERGES BY p-TEST WITH $p = \frac{1}{2} \leq 1$.

$\therefore$ THE ORIGINAL INTEGRAL MUST ALSO DIVERGE BY THE DIRECT COMPARISON THEOREM.

(b) (8 points) $\displaystyle\int_2^\infty \frac{1}{x\sqrt{x} - 1}\,dx$

LET $f(x) = \dfrac{1}{x\sqrt{x} - 1}$ AND $g(x) = \dfrac{1}{x\sqrt{x}}$.

THEN $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x\sqrt{x}}{x\sqrt{x} - 1} = \lim_{x\to\infty} \frac{1}{1 - \frac{1}{x\sqrt{x}}} = 1$

$\therefore$ SINCE $\displaystyle\int_2^\infty \frac{1}{x\sqrt{x}}\,dx = \int_2^\infty \frac{1}{x^{3/2}}\,dx$ CONVERGES BY p-TEST WITH $p = \frac{3}{2} > 1$,

THE ORIGINAL INTEGRAL ALSO CONVERGES BY THE LIMIT COMPARISON THEOREM.