

SOLUTIONS BY JOHN ADAMSKI

1. EVALUATE THE FOLLOWING INTEGRALS.

(a) $\int_1^e x^4 \ln x \, dx$ integration by parts.

$$\text{Let } u = \ln x \qquad v = \frac{1}{5} x^5$$

$$du = \frac{1}{x} dx \qquad dv = x^4 dx$$

$$\int_1^e u dv = uv \Big|_1^e - \int_1^e v du = \frac{1}{5} x^5 \ln x \Big|_1^e - \frac{1}{5} \int_1^e x^4 dx$$

$$= \frac{1}{5} \left[e^5 - 0 \right] - \frac{1}{25} x^5 \Big|_1^e = \frac{1}{5} e^5 - \frac{1}{25} \left[e^5 - 1 \right]$$

(b) $\int \cos^3(2t) \sin^5(2t) \, dt = \int (1 - \sin^2(2t)) \sin^5(2t) \cos(2t) \, dt$

$$\text{Let } u = \sin(2t) \qquad = \frac{1}{2} \int (1 - u^2) u^5 \, du$$

$$du = 2 \cos(2t) \, dt$$

$$= \frac{1}{2} \int u^5 - u^7 \, du$$

$$= \frac{1}{2} \left[\frac{1}{6} u^6 - \frac{1}{8} u^8 \right] + C$$

$$(c) \int_0^{\sqrt{2}/4} \frac{2dt}{\sqrt{1-4t^2}} \quad \text{Let } u = 2t \quad \longrightarrow \quad \int_{u=0}^{u=\sqrt{2}/4} \longrightarrow \int_{t=2(0)}^{t=2(\sqrt{2}/4)}$$

$$du = 2dt$$

$$\leadsto \int_0^{\sqrt{2}/2} \frac{1}{\sqrt{1-u^2}} du = \sin^{-1} u \Big|_0^{\sqrt{2}/2} = \sin^{-1} \frac{\sqrt{2}}{2} - \sin^{-1} 0$$

$$= \frac{\pi}{4} - 0 = \boxed{\frac{\pi}{4}}$$

$$(d) \int \frac{e^t}{e^{2t} + 3e^t + 2} dt \quad \text{Let } u = e^t$$

$$du = e^t dt$$

$$\leadsto \int \frac{1}{u^2 + 3u + 2} du = \int \frac{1}{(u+2)(u+1)} du$$

$$\text{PARTIAL FRACTIONS: } \frac{1}{(u+2)(u+1)} = \frac{A}{u+2} + \frac{B}{u+1}$$

$$\Rightarrow 1 = A(u+1) + B(u+2)$$

$$u = -2 \Rightarrow 1 = -A \Rightarrow A = -1$$

$$u = -1 \Rightarrow 1 = B \Rightarrow B = 1$$

$$\int \frac{-1}{u+2} + \frac{1}{u+1} du = -\ln|u+2| + \ln|u+1| + C$$

$$\rightarrow \boxed{\ln|e^t+1| - \ln|e^t+2| + C}$$

2. (a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{3^n + 1}$

DIRECT COMPARISON THEOREM

SINCE $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{3^n + 1} \right| \leq \sum_{n=1}^{\infty} \frac{1}{3^n}$,

AND $\sum_{n=1}^{\infty} \frac{1}{3^n}$ CONVERGES

GEOMETRIC SERIES $\sum_{n=1}^{\infty} ar^{n-1}$
 WITH $a = \frac{1}{3}$, $r = \frac{1}{3}$
 ↑
 CONVERGES FOR $|r| < 1$

THE SERIES CONVERGES ABSOLUTELY.

(b) $\sum_{n=1}^{\infty} \frac{\ln(n^2)}{n}$

DIRECT COMPARISON THEOREM

SINCE $\frac{\ln(n^2)}{n} \geq \frac{1}{n}$ FOR $n \geq 2$.

AND $\sum_{n=1}^{\infty} \frac{1}{n}$ DIVERGES (p-SERIES WITH $p=1$)

THE SERIES ALSO DIVERGES.

$$(c) \sum_{n=1}^{\infty} \frac{n!}{n^2 10^n}$$

$$\text{RATIO TEST: } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\cancel{(n+1)}n!}{(n+1)^2 \cancel{10^{n+1}}} \cdot \frac{\cancel{n^2} \cancel{10^n}}{\cancel{n!}} \right|$$

$$= \lim_{n \rightarrow \infty} \frac{n^2}{10(n+1)} = \frac{1}{10} \lim_{n \rightarrow \infty} \frac{n}{1 + \frac{1}{n}} = \infty$$

SINCE THE LIMIT IS NOT IN $[0, 1)$, THE SERIES DIVERGES.

3. THIS TOPIC HAS BEEN REMOVED FROM THE SYLLABUS.

THIS CHANGE IS NEW THIS SEMESTER, SPRING 2020.

4. (a) WE KNOW THAT THE GEOMETRIC SERIES

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \dots \text{ CONVERGES TO } \frac{a}{1-r}$$

WHEN $|r| < 1$.

SETTING $a = 1$ AND $r = -x^2$, WE HAVE

$$\frac{1}{1+x^2} = \sum_{n=1}^{\infty} 1(-x^2)^{n-1} = 1 - x^2 + x^4 - x^6 + x^8 - \dots$$

THESE ARE,

OF COURSE, THE SAME.

$$= \sum_{n=0}^{\infty} (-x^2)^n$$

(b) SINCE $\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$

WE HAVE $\tan^{-1} x = \int 1 - x^2 + x^4 - x^6 + \dots dx$

$$= C + x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots$$

SINCE $\tan^{-1} 0 = 0$, WE HAVE $C = 0$.

THUS $\tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots$

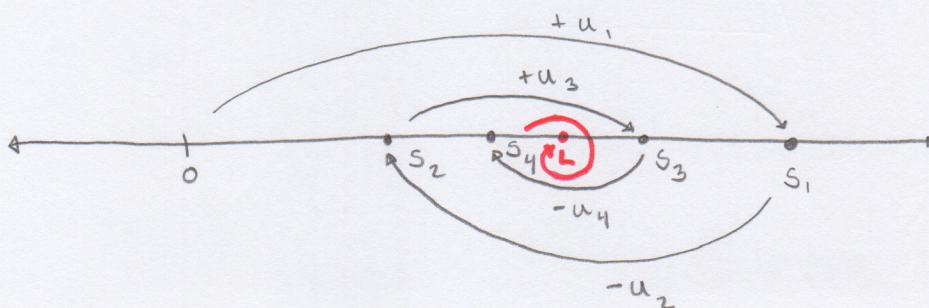
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$

(c) $\tan^{-1}(.1) = (.1) - \frac{1}{3}(.1)^3 + \frac{1}{5}(.1)^5 - \dots$

↑

NOTE THAT THIS TERM IS LESS THAN ONE THOUSANDTH.

SINCE THE SERIES IS ALTERNATING AND THE TERMS ARE DECREASING TO 0, THE PARTIAL SUM OF JUST THE FIRST 2 TERMS IS WITHIN THE ABSOLUTE VALUE OF THE NEXT TERM $\left(\left| \frac{1}{5}(.1)^5 \right| = .000002 \right)$ OF THE SUM OF THE SERIES.



Thus $\tan^{-1}(1) \approx \boxed{(1) - \frac{1}{3}(1)^3}$

5. (a) $f(x) = \sum_{n=1}^{\infty} \frac{(2x+1)^n}{n+1}$

Ratio test: $\rho(x) = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}(x)}{a_n(x)} \right|$

$= \lim_{n \rightarrow \infty} \left| \frac{(2x+1)^{\cancel{n+1}}}{n+2} \cdot \frac{n+1}{(2x+1)^{\cancel{n}}} \right|$

CONVERGES ABSOLUTELY
WHEN $\rho(x) < 1$

$= |2x+1| \lim_{n \rightarrow \infty} \left| \frac{n+1}{n+2} \right| = |2x+1| < 1$

$\Rightarrow -1 < 2x+1 < 1$

$-2 < 2x < 0$

$-1 < x < 0$

ENDPOINTS: $x = -1 : \sum_{n=1}^{\infty} \frac{(-1)^n}{n+1}$

CONVERGES BY ALT. SERIES TEST.

$x = 0 : \sum_{n=1}^{\infty} \frac{1}{n+1}$ DIVERGES (HARMONIC SERIES)

$\therefore f(x)$ converges on

$\boxed{[-1, 0)}$

$$(b) \quad f'(x) = \frac{d}{dx} \sum_{n=1}^{\infty} \frac{(2x+1)^n}{n+1}$$

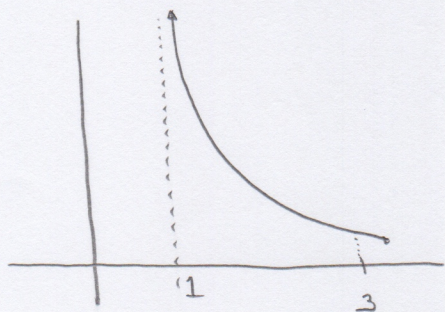
$$= \sum_{n=1}^{\infty} \frac{d}{dx} \left[\frac{(2x+1)^n}{n+1} \right]$$

$$= \sum_{n=1}^{\infty} \frac{2n(2x+1)^{n-1}}{n+1}$$

$$f'(-\frac{1}{2}) = \sum_{n=1}^{\infty} \frac{2n(0)^{n-1}}{n+1} = 1 + 0 + 0 + \dots$$

$$= \boxed{1}$$

6. (a)



$$(b) \quad \int_1^3 \frac{1}{(x-1)^{2/3}} dx = \int_0^2 u^{-2/3} du = \lim_{t \rightarrow 0^+} \int_t^2 u^{-2/3} du$$

$$= \lim_{t \rightarrow 0^+} 3u^{1/3} \Big|_t^2 = 3 \lim_{t \rightarrow 0^+} (2^{1/3} - t^{1/3})$$

$$= \boxed{3 \cdot 2^{1/3} \text{ (FINITE)}}$$

$$7. \quad x = 1 + 2t \quad y = -1 - t \quad z = 3t$$

$$xy\text{-PLANE } (z=0): 3t=0 \Rightarrow t=0$$

$$\Rightarrow (1, -1, 0)$$

$$yz\text{-PLANE } (x=0): 1+2t=0 \Rightarrow t=-\frac{1}{2}$$

$$\Rightarrow (0, -\frac{1}{2}, -\frac{3}{2})$$

$$xz\text{-PLANE } (y=0): -1-t=0 \Rightarrow t=-1$$

$$\Rightarrow (3, 0, 3)$$

8. (a) A LINE IS PARALLEL TO A PLANE IF IT IS PERPENDICULAR TO A VECTOR NORMAL TO THE PLANE

PLANE $2x + y - z = 8$ HAS NORMAL VECTOR $\vec{n} = \langle 2, 1, -1 \rangle$.

THE LINE $x = 1 - 2t$, $y = 2 + 5t$, $z = -3t$

$$\text{i.e. } \vec{r}(t) = \langle 1, 2, 0 \rangle + t \langle -2, 5, -3 \rangle$$

IS PARALLEL TO $\vec{v} = \langle -2, 5, -3 \rangle$

$\therefore$ THE LINE IS PARALLEL TO THE PLANE IF $\vec{n} \perp \vec{v}$,

$$\text{i.e. IF } \vec{n} \cdot \vec{v} = 0$$

$$\vec{n} \cdot \vec{v} = \langle 2, 1, -1 \rangle \cdot \langle -2, 5, -3 \rangle = -4 + 5 + 3 = 4 \neq 0.$$

$\therefore$ THE LINE IS NOT PARALLEL TO THE PLANE.

(b) The line of intersections will be perpendicular to both normal vectors. That is, parallel to the cross product of the normal vectors.

$$\langle 3, -6, -2 \rangle \times \langle 2, 1, -2 \rangle$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix} = \langle 12+2, -4+6, 3+12 \rangle$$
$$= \langle 14, 2, 15 \rangle$$

9. The shortest line segment from the point to the plane is perpendicular to the plane, i.e. parallel to a vector normal to the plane.

That is, a line through $(2, -3, 4)$, parallel to $\langle 1, 2, 2 \rangle$.

$$\Rightarrow \vec{r}(t) = \langle 2, -3, 4 \rangle + t \langle 1, 2, 2 \rangle$$
$$= \langle 2+t, -3+2t, 4+2t \rangle$$

Intersects the plane $x+2y+2z=13$ when

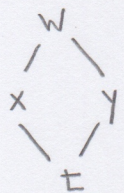
$$(2+t) + 2(-3+2t) + 2(4+2t) = 13$$

$$9t + 4 = 13 \Rightarrow t = 1$$

$$\vec{r}(1) = \langle 2+1, -3+2, 4+2 \rangle$$

$$\text{i.e. } \langle 3, -1, 6 \rangle$$

10. $w = x^2 - y^2 + z$; $x = \sinh t$, $y = \cosh t$, $z = e^{2t}$

(a)  $\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dt}$

$$\frac{dw}{dt} = 2x \cosh t - 2y \sinh t + 1 \cdot 2e^{2t}$$

$$\frac{dw}{dt} = 2 \sinh t \cosh t - 2 \cosh t \sinh t + 2e^{2t} = \boxed{2e^{2t}}$$

(b) $w = (\sinh t)^2 - (\cosh t)^2 + e^{2t}$

$$w = -1 + e^{2t}$$

$$\left[\text{IDENTITY: } \cosh^2 t - \sinh^2 t = 1 \right]$$

$$\frac{dw}{dt} = \boxed{2e^{2t}}$$

(c) $\left. \frac{dw}{dt} \right|_{t=1} = \boxed{2e^2}$