MATH 195

§2.8 ONE-TO-ONE FUNCTIONS & THEIR INVERSES

2, 4, 7-70

2. (a) 1-1, $g(x) = x^3$

(b) $\sqrt[3]{x}$

4. YES, BECAUSE IT PASSES THE HORIZONTAL LINE TEST

$f^{-1}(1) = 4$, $f^{-1}(3) = 5$

7. NO   8. YES   9. YES   10. NO   11. NO   12. YES

(APPLY HORIZONTAL LINE TEST)

13. YES, $f^{-1}(x) = \dfrac{x-4}{-2}$

14. YES, $f^{-1}(x) = \dfrac{x+2}{3}$

15. YES, $g^{-1}(x) = x^2$, $x \geq 0$

16. NO, $g(-a) = g(a)$ *
FOR ALL $a \in \mathbb{R}$

17. NO, $h(0) = h(2)$

18. YES, $h^{-1}(x) = \sqrt[3]{x-8}$

19. NO, $f(-a) = f(a)$ *
FOR ALL $a \in \mathbb{R}$

20. YES, $f^{-1}(x) = \sqrt[4]{x-5}$, $0 \leq x \leq 21$

21. YES, $r^{-1}(t) = \sqrt[6]{t+3}$, $0 \leq t \leq 5^6 - 3$

22. NO, $r(-a) = r(a)$ *
FOR ALL $a \in \mathbb{R}$

23. NO, $f(-a) = f(a)$ *
FOR ALL $a \in \mathbb{R}$

24. YES, $f^{-1}(x) = \dfrac{1}{x}$

* SO THESE FUNCTIONS ARE EVEN!

25. (a) 2 (b) 3

26. (a) 5 (b) 4

27. (a) $f(x) = 5 - 2x = 3$ ← FOR WHAT INPUT (x)

$-2x = -2$ IS THE OUTPUT $(f(x))$

$x = \boxed{1}$ EQUAL TO 3? SOLVE.

28. $g(x) = x^2 + 4x = 5$

$x^2 + 4x - 5 = 0$

$(x+5)(x-1) = 0$

$x = -5$, $x = 1$. SINCE $x \geq -2$, WE HAVE $\boxed{x = 1}$

29. (a) $f^{-1}(2) = \boxed{6}$ (b) $f^{-1}(5) = \boxed{2}$ (c) $f^{-1}(6) = \boxed{0}$

$(6, 2)$ $(2, 5)$ $(0, 6)$

30. (a) $g^{-1}(2) = \boxed{4}$ (b) $g^{-1}(5) = \boxed{7}$ (c) $g^{-1}(6) = \boxed{8}$

$(4, 2)$ $(7, 5)$ $(8, 6)$

31. $f^{-1}(5) = \boxed{4}$

32. $f^{-1}(0) = \boxed{5}$

33. $f^{-1}(f(1)) = \boxed{1}$

34. $f(f^{-1}(6)) = \boxed{6}$

35. $f^{-1}(f^{-1}(1)) = f^{-1}(6) = \boxed{2}$

36. $f^{-1}(f^{-1}(0)) = f^{-1}(5) = \boxed{4}$

37. $f(g(x)) = (x+6) - 6 = x$ ✓

$g(f(x)) = (x-6) + 6 = x$ ✓

38. $f(g(x)) = 3\left(\frac{x}{3}\right) = x$ ✓

$g(f(x)) = \frac{(3x)}{3} = x$ ✓

39. $f(g(x)) = 3\left(\frac{x-4}{3}\right) + 4 = (x-4) + 4 = x$ ✓

$g(f(x)) = \frac{(3x+4)-4}{3} = \frac{3x}{3} = x$ ✓

40. $f(g(x)) = 2 - 5\left(\frac{2-x}{5}\right) = 2 - (2-x) = x$ ✓

$g(f(x)) = \frac{2-(2-5x)}{5} = \frac{5x}{5} = x$ ✓

41. $f(g(x)) = g(f(x)) = \frac{1}{\frac{1}{x}} = x$ ✓

42. $f(g(x)) = \left(\sqrt[5]{x}\right)^5 = x$ ✓

$g(f(x)) = \sqrt[5]{x^5} = x$ ✓

43. $f(g(x)) = \left(\sqrt{x+9}\right)^2 - 9 = x, \quad x \geq -9$

$g(f(x)) = \sqrt{(x^2-9)+9} = \sqrt{x^2} = x, \quad x \geq 0$

44. $f(g(x)) = \left((x-1)^{1/3}\right)^3 + 1 = x$ ✓

$g(f(x)) = \left((x^3+1)-1\right)^{1/3} = x$ ✓

45. $f(g(x)) = \dfrac{1}{\left(\frac{1}{x}+1\right)-1} = \dfrac{1}{\frac{1}{x}} = x$ ✓

$g(f(x)) = \dfrac{1}{\frac{1}{x-1}} + 1 = x - 1 + 1 = x$ ✓

46. $f(g(x)) = \sqrt{4 - \left(\sqrt{4-x^2}\right)^2} = \sqrt{4-(4-x^2)} = \sqrt{x^2} = x$

$g(f(x)) = f(g(x)) = x$ ✓

47. $f(g(x)) = \dfrac{\frac{2x+2}{x-1} + 2}{\frac{2x+2}{x-1} - 2} = \dfrac{2x+2+2(x-1)}{2x+2-2(x-1)}$

$= \dfrac{4x}{4} = x$ ✓

$g(f(x)) = \dfrac{2\left(\frac{x+2}{x-2}\right) + 2}{\frac{x+2}{x-2} - 1} = \dfrac{2(x+2)+2(x-2)}{x+2-(x-2)}$

$= \dfrac{4x}{4} = x$ ✓

48. $f(g(x)) = \dfrac{\dfrac{5+4x}{1-3x} - 5}{3\left(\dfrac{5+4x}{1-3x}\right) + 4} = \dfrac{5+4x-5(1-3x)}{3(5+4x)+4(1-3x)}$

$= \dfrac{19x}{19} = x$ ✓

$g(f(x)) = \dfrac{5 + 4\left(\dfrac{x-5}{3x+4}\right)}{1 - 3\left(\dfrac{x-5}{3x+4}\right)} = \dfrac{5(3x+4)+4(x-5)}{(3x+4)-3(x-5)}$

$= \dfrac{19x}{19} = x$ ✓

49. $y = 3x + 5$

$y - 5 = 3x$

$\dfrac{y-5}{3} = x \rightarrow \boxed{f^{-1}(x) = \dfrac{x-5}{3}}$

50. $y = 7 - 5x$

$y - 7 = -5x$

$\dfrac{y-7}{-5} = x \rightarrow \boxed{f^{-1}(x) = \dfrac{7-x}{5}}$

51. $y = 5 - 4x^3$

$y - 5 = -4x^3$

$\dfrac{y-5}{-4} = x^3$

$\sqrt[3]{\dfrac{y-5}{-4}} = x \rightarrow \boxed{f^{-1}(x) = \sqrt[3]{\dfrac{5-x}{4}}}$

52. $y = 3x^3 + 8$

$$\boxed{f^{-1}(x) = \sqrt[3]{\frac{x-8}{3}}}$$

53. $y = \dfrac{1}{x+2}$

$$\frac{1}{y} = x + 2$$

$$\frac{1}{y} - 2 = x \rightarrow \boxed{f^{-1}(x) = \frac{1}{x} - 2}$$

54. $y = \dfrac{x-2}{x+2}$

$$xy + 2y = x - 2$$

$$x(y-1) = -2y - 2$$

$$x = \frac{-2y-2}{y-1}$$

$$\boxed{f^{-1}(x) = \frac{2y+2}{1-y}}$$

55. $y = \dfrac{x}{x+4} \rightarrow xy + 4y = x$

$$x(y-1) = -4y$$

$$x = \frac{-4y}{y-1} \rightarrow \boxed{f^{-1}(x) = \frac{4x}{1-x}}$$

56. $y = \dfrac{3x}{x-2}$

$$xy - 2y = 3x$$

$$x(y-3) = 2y$$

$$x = \frac{2y}{y-3} \rightarrow f^{-1}(x) = \frac{2x}{x-3}$$

57. $y = \dfrac{2x+5}{x-7}$

$$xy - 7y = 2x + 5$$

$$x(y-2) = 7y + 5$$

$$x = \frac{7y+5}{y-2} \rightarrow \boxed{f^{-1}(x) = \frac{7x+5}{x-2}}$$

58. $y = \dfrac{4x-2}{3x+1}$

$$3xy + y = 4x - 2$$

$$x(3y-4) = -y - 2$$

$$x = \frac{-y-2}{3y-4} \rightarrow \boxed{f^{-1}(x) = \frac{x+2}{4-3x}}$$

59. $y = \dfrac{2x+3}{1-5x}$

$y - 5xy = 2x + 3$

$x(-5y-2) = -y+3$

$x = \dfrac{-y+3}{-5y-2} \rightarrow \boxed{f^{-1}(x) = \dfrac{y-3}{5y+2}}$

60. $y = \dfrac{3-4x}{8x-1}$

$8xy - y = 3 - 4x$

$x(8y+4) = y+3$

$x = \dfrac{y+3}{8y+4} \rightarrow \boxed{f^{-1}(x) = \dfrac{x+3}{8x+4}}$

61. $y = 4 - x^2, \quad x \geq 0$

$x^2 = 4 - y$

$x = \pm\sqrt{4-y} \rightarrow \boxed{f^{-1}(x) = \sqrt{4-y}}$

62. $y = x^2 + x$

$y = \left(x + \frac{1}{2}\right)^2 - \frac{1}{4}$

$y + \frac{1}{4} = \left(x+\frac{1}{2}\right)^2$

$\pm\sqrt{y+\frac{1}{4}} = \underbrace{x + \frac{1}{2} \geq 0}_{\left(x \geq -\frac{1}{2}\right)}$

$x = \sqrt{y+\frac{1}{4}} - \frac{1}{2} \rightarrow \boxed{f^{-1}(x) = \sqrt{x+\frac{1}{4}} - \frac{1}{2}}$

63. $y = x^6, \quad x \geq 0$

$\pm\sqrt[6]{y} = x \rightarrow \boxed{f^{-1}(x) = \sqrt[6]{x}}$

64. $y = \dfrac{1}{x^2}$

$x^2 = \dfrac{1}{y} \rightarrow \overbrace{x = \pm\sqrt{\dfrac{1}{y}} \geq 0}^{x \geq 0}$

$\boxed{f^{-1}(x) = \sqrt{\dfrac{1}{x}} = \dfrac{1}{\sqrt{x}}}$

65. $y = \dfrac{2 - x^3}{5}$

$5y = 2 - x^3$

$5y - 2 = x^3$

$\sqrt[3]{5y - 2} = x$

$f^{-1}(x) = \sqrt[3]{5x - 2}$

66. $y = (x^5 - 6)^7$

$\sqrt[7]{y} = x^5 - 6$

$\sqrt[7]{y} + 6 = x^5$

$\sqrt[5]{\sqrt[7]{y} + 6} = x \longrightarrow \boxed{f^{-1}(x) = \sqrt[5]{\sqrt[7]{x} + 6}}$

67. $y = \sqrt{5 + 8x}$ Note: $y \geq 0$

$y^2 - 5 = 8x$

$\dfrac{y^2 - 5}{8} = x \quad , \quad y \geq 0$

$$\boxed{f^{-1}(x) = \frac{x^2 - 5}{8} \quad , \quad x \geq 0}$$

68. $y = 2 + \sqrt{3 + x}$ Note: $y \geq 2$

$y - 2 = \sqrt{3 + x}$

$(y - 2)^2 = 3 + x$

$(y - 2)^2 - 3 = x \quad , \quad y \geq 2$

$$\boxed{f^{-1}(x) = (x - 2)^2 - 3 \quad , \quad x \geq 2}$$

69. $y = 2 + \sqrt[3]{x}$

$y - 2 = \sqrt[3]{x}$

$(y - 2)^3 = x$

$$\boxed{f^{-1}(x) = (x - 2)^3}$$

70. $y = \sqrt{4 - x^2} \quad , \quad 0 \leq x \leq 2$

(Note also, $0 \leq y \leq 2$)

$y^2 = 4 - x^2$

$y^2 - 4 = -x^2$

$4 - y^2 = x^2$

$\pm\sqrt{4 - y^2} = x \geq 0$

$$\boxed{f^{-1}(x) = \sqrt{4 - x^2} \quad , \quad 0 \leq x \leq 2}$$