2.6 TRANSFORMATIONS OF FUNCTIONS

■ Vertical Shifting ■ Horizontal Shifting ■ Reflecting Graphs ■ Vertical Stretching and Shrinking ■ Horizontal Stretching and Shrinking ■ Even and Odd Functions

In this section we study how certain transformations of a function affect its graph. This will give us a better understanding of how to graph functions. The transformations that we study are shifting, reflecting, and stretching.

Vertical Shifting

Adding a constant to a function shifts its graph vertically: upward if the constant is positive and downward if it is negative.

In general, suppose we know the graph of $y = f(x)$. How do we obtain from it the graphs of

$$y = f(x) + c \qquad \text{and} \qquad y = f(x) - c \qquad (c > 0)$$

Recall that the graph of the function f is the same as the graph of the equation $y = f(x)$.

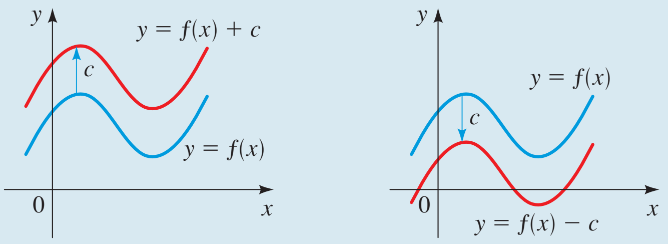
The y-coordinate of each point on the graph of $y = f(x) + c$ is c units above the y-coordinate of the corresponding point on the graph of $y = f(x)$. So we obtain the graph of $y = f(x) + c$ simply by shifting the graph of $y = f(x)$ upward c units. Similarly, we obtain the graph of $y = f(x) - c$ by shifting the graph of $y = f(x)$ downward c units.

VERTICAL SHIFTS OF GRAPHS

Suppose $c > 0$.

To graph $y = f(x) + c$, shift the graph of $y = f(x)$ upward c units.

To graph $y = f(x) - c$, shift the graph of $y = f(x)$ downward c units.



EXAMPLE 1 ■ Vertical Shifts of Graphs

Use the graph of $f(x) = x^2$ to sketch the graph of each function.

(a) $g(x) = x^2 + 3$ **(b)** $h(x) = x^2 - 2$

SOLUTION The function $f(x) = x^2$ was graphed in Example 1(a), Section 2.2. It is sketched again in Figure 1.

(a) Observe that

$$g(x) = x^2 + 3 = f(x) + 3$$

So the y-coordinate of each point on the graph of g is 3 units above the corresponding point on the graph of f. This means that to graph g, we shift the graph of f upward 3 units, as in Figure 1.

(b) Similarly, to graph h we shift the graph of f downward 2 units, as shown in Figure 1.

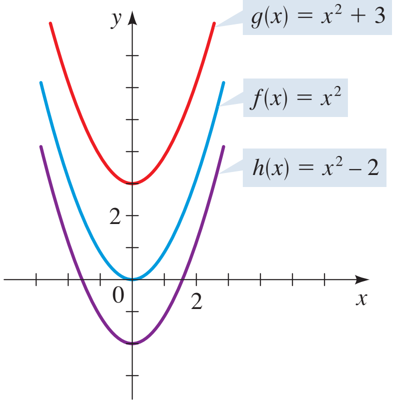


FIGURE 1

Now Try Exercises 29 and 31

Horizontal Shifting

Suppose that we know the graph of $y = f(x)$. How do we use it to obtain the graphs of

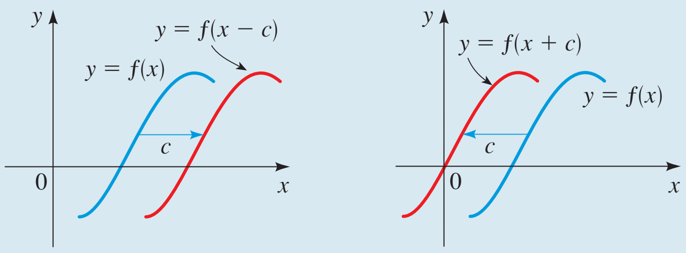
$$y = f(x + c) \qquad \text{and} \qquad y = f(x - c) \qquad (c > 0)$$

The value of $f(x - c)$ at x is the same as the value of $f(x)$ at $x - c$. Since $x - c$ is c units to the left of x, it follows that the graph of $y = f(x - c)$ is just the graph of $y = f(x)$ shifted to the right c units. Similar reasoning shows that the graph of $y = f(x + c)$ is the graph of $y = f(x)$ shifted to the left c units. The following box summarizes these facts.

HORIZONTAL SHIFTS OF GRAPHS

Suppose $c > 0$.

To graph $y = f(x - c)$, shift the graph of $y = f(x)$ to the right c units.
To graph $y = f(x + c)$, shift the graph of $y = f(x)$ to the left c units.



EXAMPLE 2 ■ Horizontal Shifts of Graphs

Use the graph of $f(x) = x^2$ to sketch the graph of each function.

(a) $g(x) = (x + 4)^2$ **(b)** $h(x) = (x - 2)^2$

SOLUTION

(a) To graph g, we shift the graph of f to the left 4 units.

(b) To graph h, we shift the graph of f to the right 2 units.

The graphs of g and h are sketched in Figure 2.

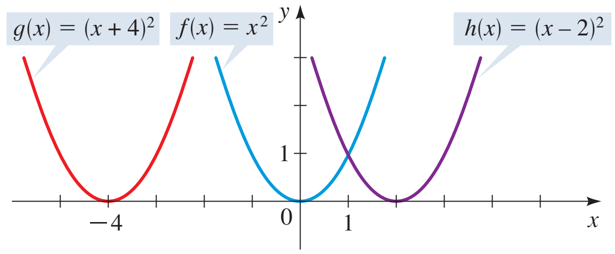


FIGURE 2

Now Try Exercises 33 and 35 ■

EXAMPLE 3 ■ Combining Horizontal and Vertical Shifts

Sketch the graph of $f(x) = \sqrt{x - 3} + 4$.

SOLUTION We start with the graph of $y = \sqrt{x}$ (Example 1(c), Section 2.2) and shift it to the right 3 units to obtain the graph of $y = \sqrt{x - 3}$. Then we shift the resulting graph upward 4 units to obtain the graph of $f(x) = \sqrt{x - 3} + 4$ shown in Figure 3.

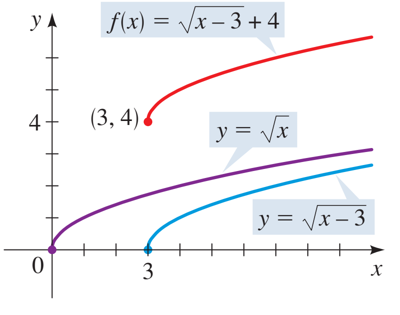


FIGURE 3

Now Try Exercise 45 ■

■ Reflecting Graphs

Suppose we know the graph of $y = f(x)$. How do we use it to obtain the graphs of $y = -f(x)$ and $y = f(-x)$? The y-coordinate of each point on the graph of $y = -f(x)$ is simply the negative of the y-coordinate of the corresponding point on the graph of $y = f(x)$. So the desired graph is the reflection of the graph of $y = f(x)$ in the x-axis. On the other hand, the value of $y = f(-x)$ at x is the same as the value of $y = f(x)$ at

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DISCOVERY PROJECT

Transformation Stories

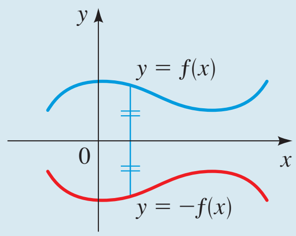
If a real-world situation, or "story," is modeled by a function, how does transforming the function change the story? For example, if the distance traveled on a road trip is modeled by a function, then how does shifting or stretching the function change the story of the trip? How does changing the story of the trip transform the function that models the trip? In this project we explore some real-world stories and transformations of these stories. You can find the project at **www.stewartmath.com**.

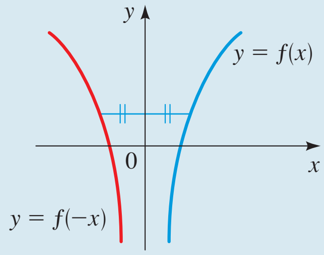
$-x$, so the desired graph here is the reflection of the graph of $y = f(x)$ in the y-axis. The following box summarizes these observations.

REFLECTING GRAPHS

To graph $y = -f(x)$, reflect the graph of $y = f(x)$ in the x-axis.

To graph $y = f(-x)$, reflect the graph of $y = f(x)$ in the y-axis.





EXAMPLE 4 ■ Reflecting Graphs

Sketch the graph of each function.

(a) $f(x) = -x^2$ **(b)** $g(x) = \sqrt{-x}$

SOLUTION

(a) We start with the graph of $y = x^2$. The graph of $f(x) = -x^2$ is the graph of $y = x^2$ reflected in the x-axis (see Figure 4).

(b) We start with the graph of $y = \sqrt{x}$ (Example 1(c) in Section 2.2). The graph of $g(x) = \sqrt{-x}$ is the graph of $y = \sqrt{x}$ reflected in the y-axis (see Figure 5). Note that the domain of the function $g(x) = \sqrt{-x}$ is $\{x \mid x \leq 0\}$.

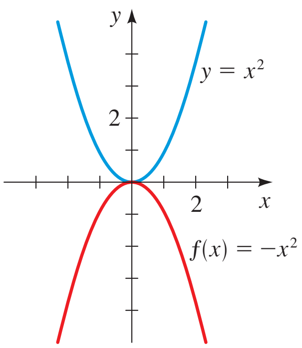


FIGURE 4

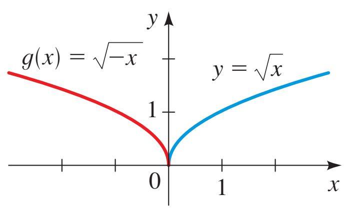


FIGURE 5

Now Try Exercises 37 and 39 ■

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RENÉ DESCARTES (1596–1650) was born in the town of La Haye in southern France. From an early age Descartes liked mathematics because of "the certainty of its results and the clarity of its reasoning." He believed that to arrive at truth, one must begin by doubting everything, including one's own existence; this led him to formulate perhaps the best-known sentence in all of philosophy: "I think, therefore I am." In his book *Discourse on Method* he described what is now called the Cartesian plane. This idea of combining algebra and geometry enabled mathematicians for the first time to graph functions and thus "see" the equations they were studying. The philosopher John Stuart Mill called this invention "the greatest single step ever made in the progress of the exact sciences." Descartes liked to get up late and spend the morning in bed thinking and writing. He invented the coordinate plane while lying in bed watching a fly crawl on the ceiling, reasoning that he could describe the exact location of the fly by knowing its distance from two perpendicular walls. In 1649 Descartes became the tutor of Queen Christina of Sweden. She liked her lessons at 5 o'clock in the morning, when, she said, her mind was sharpest. However, the change from his usual habits and the ice-cold library where they studied proved too much for Descartes. In February 1650, after just two months of this, he caught pneumonia and died.

Vertical Stretching and Shrinking

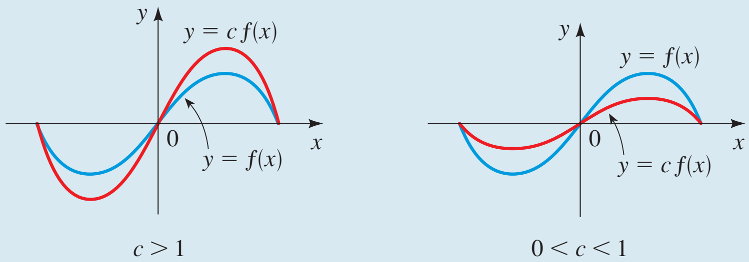
Suppose we know the graph of $y = f(x)$. How do we use it to obtain the graph of $y = cf(x)$? The y-coordinate of $y = cf(x)$ at x is the same as the corresponding y-coordinate of $y = f(x)$ multiplied by c. Multiplying the y-coordinates by c has the effect of vertically stretching or shrinking the graph by a factor of c (if $c > 0$).

VERTICAL STRETCHING AND SHRINKING OF GRAPHS

To graph $y = cf(x)$:

If $c > 1$, stretch the graph of $y = f(x)$ vertically by a factor of c.

If $0 < c < 1$, shrink the graph of $y = f(x)$ vertically by a factor of c.



EXAMPLE 5 ■ Vertical Stretching and Shrinking of Graphs

Use the graph of $f(x) = x^2$ to sketch the graph of each function.

(a) $g(x) = 3x^2$ **(b)** $h(x) = \frac{1}{3}x^2$

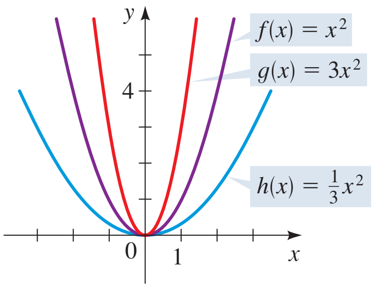


FIGURE 6

SOLUTION

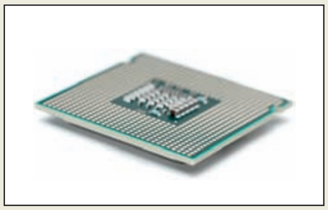
(a) The graph of g is obtained by multiplying the y-coordinate of each point on the graph of f by 3. That is, to obtain the graph of g, we stretch the graph of f vertically by a factor of 3. The result is the narrowest parabola in Figure 6.

(b) The graph of h is obtained by multiplying the y-coordinate of each point on the graph of f by $\frac{1}{3}$. That is, to obtain the graph of h, we shrink the graph of f vertically by a factor of $\frac{1}{3}$. The result is the widest parabola in Figure 6.

Now Try Exercises 41 and 43 ■

We illustrate the effect of combining shifts, reflections, and stretching in the following example.

Mathematics in the Modern World



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Computers

For centuries machines have been designed to perform specific tasks. For example, a washing machine washes clothes, a weaving machine weaves cloth, an adding machine adds numbers, and so on. The computer has changed all that.

The computer is a machine that does nothing—until it is given instructions on what to do. So your computer can play games, draw pictures, or calculate π to a million decimal places; it all depends on what program (or instructions) you give the computer. The computer can do all this because it is able to accept instructions and logically change those instructions based on incoming data. This versatility makes computers useful in nearly every aspect of human endeavor.

The idea of a computer was described theoretically in the 1940s by the mathematician Allan Turing (see page 118) in what he called a *universal machine*. In 1945 the mathematician John Von Neumann, extending Turing's ideas, built one of the first electronic computers.

Mathematicians continue to develop new theoretical bases for the design of computers. The heart of the computer is the "chip," which is capable of processing logical instructions. To get an idea of the chip's complexity, consider that the Pentium chip has over 3.5 million logic circuits!

EXAMPLE 6 ■ Combining Shifting, Stretching, and Reflecting

Sketch the graph of the function $f(x) = 1 - 2(x - 3)^2$.

SOLUTION Starting with the graph of $y = x^2$, we first shift to the right 3 units to get the graph of $y = (x - 3)^2$. Then we reflect in the x-axis and stretch by a factor of 2 to get the graph of $y = -2(x - 3)^2$. Finally, we shift upward 1 unit to get the graph of $f(x) = 1 - 2(x - 3)^2$ shown in Figure 7.

Note that the shifts and stretches follow the normal order of operations when evaluating the function. In particular, the upward shift must be performed *last*.

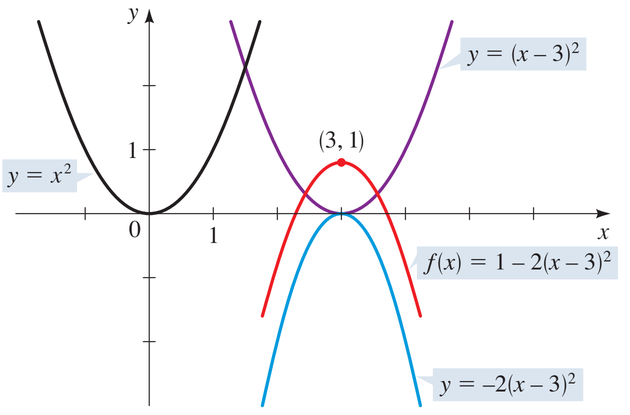


FIGURE 7

Now Try Exercise 47

■ Horizontal Stretching and Shrinking

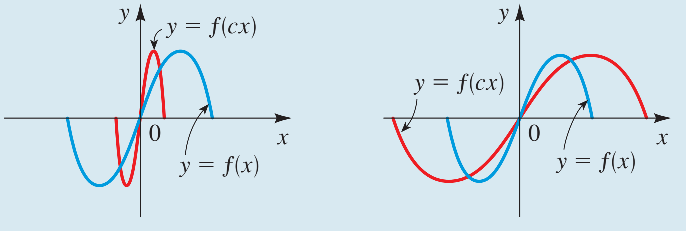
Now we consider horizontal shrinking and stretching of graphs. If we know the graph of $y = f(x)$, then how is the graph of $y = f(cx)$ related to it? The y-coordinate of $y = f(cx)$ at x is the same as the y-coordinate of $y = f(x)$ at cx. Thus the x-coordinates in the graph of $y = f(x)$ correspond to the x-coordinates in the graph of $y = f(cx)$ multiplied by c. Looking at this the other way around, we see that the x-coordinates in the graph of $y = f(cx)$ are the x-coordinates in the graph of $y = f(x)$ multiplied by $1/c$. In other words, to change the graph of $y = f(x)$ to the graph of $y = f(cx)$, we must shrink (or stretch) the graph horizontally by a factor of $1/c$ (if $c > 0$), as summarized in the following box.

HORIZONTAL SHRINKING AND STRETCHING OF GRAPHS

To graph $y = f(cx)$:

If $c > 1$, shrink the graph of $y = f(x)$ horizontally by a factor of $1/c$.

If $0 < c < 1$, stretch the graph of $y = f(x)$ horizontally by a factor of $1/c$.



EXAMPLE 7 ■ Horizontal Stretching and Shrinking of Graphs

The graph of $y = f(x)$ is shown in Figure 8. Sketch the graph of each function.

(a) $y = f(2x)$ **(b)** $y = f\left(\frac{1}{2}x\right)$

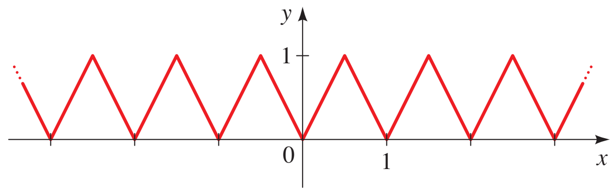


FIGURE 8 $y = f(x)$

SOLUTION Using the principles described on page 203, we **(a)** *shrink* the graph horizontally by the factor $\frac{1}{2}$ to obtain the graph in Figure 9, and **(b)** *stretch* the graph horizontally by the factor 2 to obtain the graph in Figure 10.

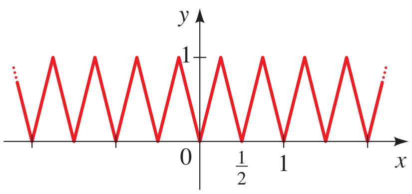


FIGURE 9 $y = f(2x)$

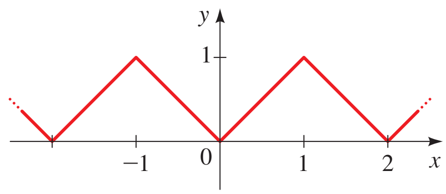


FIGURE 10 $y = f\left(\frac{1}{2}x\right)$

Now Try Exercise 71 ■

■ Even and Odd Functions

If a function f satisfies $f(-x) = f(x)$ for every number x in its domain, then f is called an **even function**. For instance, the function $f(x) = x^2$ is even because

$$f(-x) = (-x)^2 = (-1)^2x^2 = x^2 = f(x)$$

The graph of an even function is symmetric with respect to the y-axis (see Figure 11). This means that if we have plotted the graph of f for $x \geq 0$, then we can obtain the entire graph simply by reflecting this portion in the y-axis.

If f satisfies $f(-x) = -f(x)$ for every number x in its domain, then f is called an **odd function**. For example, the function $f(x) = x^3$ is odd because

$$f(-x) = (-x)^3 = (-1)^3x^3 = -x^3 = -f(x)$$

The graph of an odd function is symmetric about the origin (see Figure 12). If we have plotted the graph of f for $x \geq 0$, then we can obtain the entire graph by rotating this portion through 180° about the origin. (This is equivalent to reflecting first in the x-axis and then in the y-axis.)

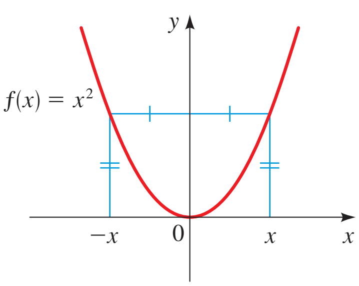


FIGURE 11 $f(x) = x^2$ is an even function.

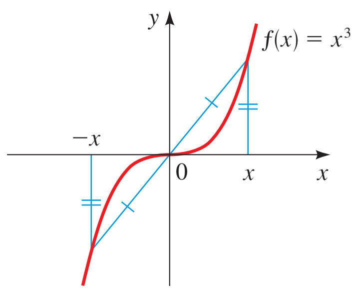


FIGURE 12 $f(x) = x^3$ is an odd function.

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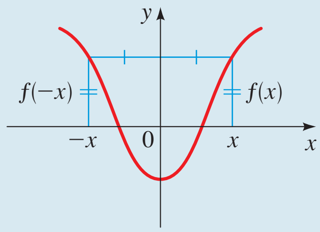
SONYA KOVALEVSKY (1850–1891) is considered the most important woman mathematician of the 19th century. She was born in Moscow to an aristocratic family. While a child, she was exposed to the principles of calculus in a very unusual fashion: Her bedroom was temporarily wallpapered with the pages of a calculus book. She later wrote that she "spent many hours in front of that wall, trying to understand it." Since Russian law forbade women from studying in universities, she entered a marriage of convenience, which allowed her to travel to Germany and obtain a doctorate in mathematics from the University of Göttingen. She eventually was awarded a full professorship at the University of Stockholm, where she taught for eight years before dying in an influenza epidemic at the age of 41. Her research was instrumental in helping to put the ideas and applications of functions and calculus on a sound and logical foundation. She received many accolades and prizes for her research work.

EVEN AND ODD FUNCTIONS

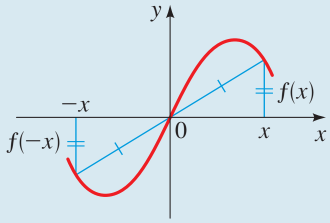
Let f be a function.

f is **even** if $f(-x) = f(x)$ for all x in the domain of f.

f is **odd** if $f(-x) = -f(x)$ for all x in the domain of f.



The graph of an even function is symmetric with respect to the y-axis.



The graph of an odd function is symmetric with respect to the origin.

EXAMPLE 8 ■ Even and Odd Functions

Determine whether the functions are even, odd, or neither even nor odd.

(a) $f(x) = x^5 + x$

(b) $g(x) = 1 - x^4$

(c) $h(x) = 2x - x^2$

SOLUTION

(a) $$\begin{aligned} f(-x) &= (-x)^5 + (-x) \\ &= -x^5 - x = -(x^5 + x) \\ &= -f(x) \end{aligned}$$

Therefore f is an odd function.

(b) $g(-x) = 1 - (-x)^4 = 1 - x^4 = g(x)$

So g is even.

(c) $h(-x) = 2(-x) - (-x)^2 = -2x - x^2$

Since $h(-x) \neq h(x)$ and $h(-x) \neq -h(x)$, we conclude that h is neither even nor odd.

Now Try Exercises 83, 85, and 87 ■

The graphs of the functions in Example 8 are shown in Figure 13. The graph of f is symmetric about the origin, and the graph of g is symmetric about the y-axis. The graph of h is not symmetric about either the y-axis or the origin.

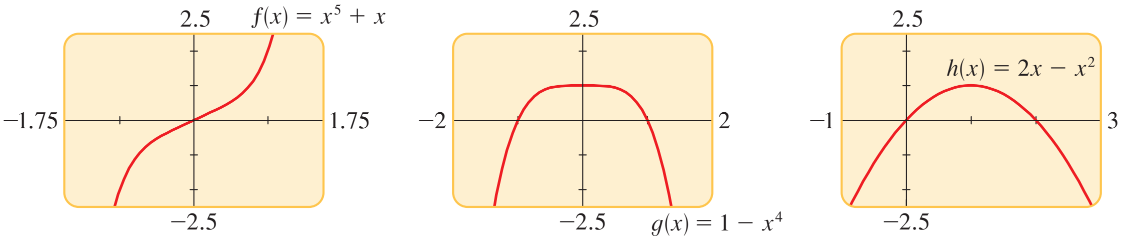


FIGURE 13 (a) (b) (c)

2.6 EXERCISES

CONCEPTS

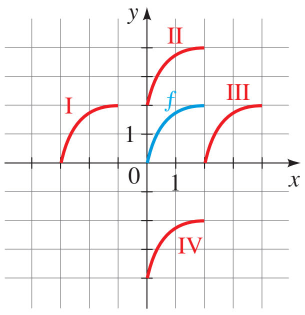
1–2 ■ Fill in the blank with the appropriate direction (left, right, up, or down).

1. **(a)** The graph of $y = f(x) + 3$ is obtained from the graph of $y = f(x)$ by shifting ________ 3 units.
(b) The graph of $y = f(x + 3)$ is obtained from the graph of $y = f(x)$ by shifting ________ 3 units.

2. **(a)** The graph of $y = f(x) - 3$ is obtained from the graph of $y = f(x)$ by shifting ________ 3 units.
(b) The graph of $y = f(x - 3)$ is obtained from the graph of $y = f(x)$ by shifting ________ 3 units.

3. Fill in the blank with the appropriate axis (x-axis or y-axis).
(a) The graph of $y = -f(x)$ is obtained from the graph of $y = f(x)$ by reflecting in the ________.
(b) The graph of $y = f(-x)$ is obtained from the graph of $y = f(x)$ by reflecting in the ________.

4. A graph of a function f is given. Match each equation with one of the graphs labeled I–IV.
(a) $f(x) + 2$ **(b)** $f(x + 3)$
(c) $f(x - 2)$ **(d)** $f(x) - 4$



5. If a function f is an even function, then what type of symmetry does the graph of f have?

6. If a function f is an odd function, then what type of symmetry does the graph of f have?

SKILLS

7–18 ■ **Describing Transformations** Suppose the graph of f is given. Describe how the graph of each function can be obtained from the graph of f.

7. **(a)** $f(x) - 1$ **(b)** $f(x - 2)$

8. **(a)** $f(x + 5)$ **(b)** $f(x) + 4$

9. **(a)** $f(-x)$ **(b)** $3f(x)$

10. **(a)** $-f(x)$ **(b)** $\frac{1}{3}f(x)$

11. **(a)** $y = f(x - 5) + 2$ **(b)** $y = f(x + 1) - 1$

12. **(a)** $y = f(x + 3) + 2$ **(b)** $y = f(x - 7) - 3$

13. **(a)** $y = -f(x) + 5$ **(b)** $y = 3f(x) - 5$

14. **(a)** $1 - f(-x)$ **(b)** $2 - \frac{1}{5}f(x)$

15. **(a)** $2f(x + 5) - 1$ **(b)** $\frac{1}{4}f(x - 3) + 5$

16. **(a)** $\frac{1}{3}f(x - 2) + 5$ **(b)** $4f(x + 1) + 3$

17. **(a)** $y = f(4x)$ **(b)** $y = f(\frac{1}{4}x)$

18. **(a)** $y = f(2x) - 1$ **(b)** $y = 2f(\frac{1}{2}x)$

19–22 ■ **Describing Transformations** Explain how the graph of g is obtained from the graph of f.

19. **(a)** $f(x) = x^2$, $g(x) = (x + 2)^2$
(b) $f(x) = x^2$, $g(x) = x^2 + 2$

20. **(a)** $f(x) = x^3$, $g(x) = (x - 4)^3$
(b) $f(x) = x^3$, $g(x) = x^3 - 4$

21. **(a)** $f(x) = |x|$, $g(x) = |x + 2| - 2$
(b) $f(x) = |x|$, $g(x) = |x - 2| + 2$

22. **(a)** $f(x) = \sqrt{x}$, $g(x) = -\sqrt{x} + 1$
(b) $f(x) = \sqrt{x}$, $g(x) = \sqrt{-x} + 1$

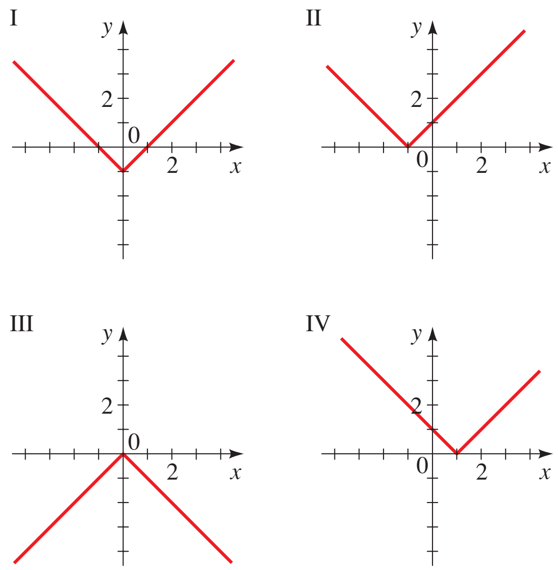
23. **Graphing Transformations** Use the graph of $y = x^2$ in Figure 4 to graph the following.
(a) $g(x) = x^2 + 1$ **(b)** $g(x) = (x - 1)^2$
(c) $g(x) = -x^2$ **(d)** $g(x) = (x - 1)^2 + 3$

24. **Graphing Transformations** Use the graph of $y = \sqrt{x}$ in Figure 5 to graph the following.
(a) $g(x) = \sqrt{x - 2}$ **(b)** $g(x) = \sqrt{x} + 1$
(c) $g(x) = \sqrt{x + 2} + 2$ **(d)** $g(x) = -\sqrt{x} + 1$

25–28 ■ **Identifying Transformations** Match the graph with the function. (See the graph of $y = |x|$ on page 96.)

25. $y = |x + 1|$ **26.** $y = |x - 1|$

27. $y = |x| - 1$ **28.** $y = -|x|$



29–52 ■ Graphing Transformations Sketch the graph of the function, not by plotting points, but by starting with the graph of a standard function and applying transformations.

29. $f(x) = x^2 + 3$

30. $f(x) = x^2 - 4$

31. $f(x) = |x| - 1$

32. $f(x) = \sqrt{x} + 1$

33. $f(x) = (x - 5)^2$

34. $f(x) = (x + 1)^2$

35. $f(x) = |x + 2|$

36. $f(x) = \sqrt{x - 4}$

37. $f(x) = -x^3$

38. $f(x) = -|x|$

39. $y = \sqrt[4]{-x}$

40. $y = \sqrt[3]{-x}$

41. $y = \frac{1}{4}x^2$

42. $y = -5\sqrt{x}$

43. $y = 3|x|$

44. $y = \frac{1}{2}|x|$

45. $y = (x - 3)^2 + 5$

46. $y = \sqrt{x + 4} - 3$

47. $y = 3 - \frac{1}{2}(x - 1)^2$

48. $y = 2 - \sqrt{x + 1}$

49. $y = |x + 2| + 2$

50. $y = 2 - |x|$

51. $y = \frac{1}{2}\sqrt{x + 4} - 3$

52. $y = 3 - 2(x - 1)^2$

53–62 ■ Finding Equations for Transformations A function f is given, and the indicated transformations are applied to its graph (in the given order). Write an equation for the final transformed graph.

53. $f(x) = x^2$; shift downward 3 units

54. $f(x) = x^3$; shift upward 5 units

55. $f(x) = \sqrt{x}$; shift 2 units to the left

56. $f(x) = \sqrt[3]{x}$; shift 1 unit to the right

57. $f(x) = |x|$; shift 2 units to the left and shift downward 5 units

58. $f(x) = |x|$; reflect in the x-axis, shift 4 units to the right, and shift upward 3 units.

59. $f(x) = \sqrt[4]{x}$; reflect in the y-axis and shift upward 1 unit

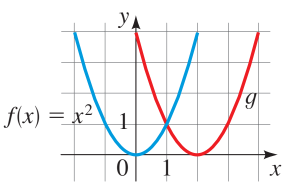
60. $f(x) = x^2$; shift 2 units to the left and reflect in the x-axis

61. $f(x) = x^2$; stretch vertically by a factor of 2, shift downward 2 units, and shift 3 units to the right

62. $f(x) = |x|$; shrink vertically by a factor of $\frac{1}{2}$, shift to the left 1 unit, and shift upward 3 units

63–68 ■ Finding Formulas for Transformations The graphs of f and g are given. Find a formula for the function g.

63.



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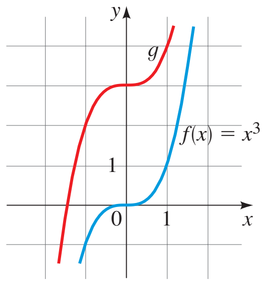

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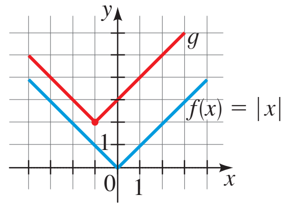

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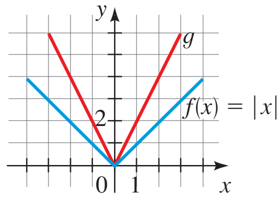

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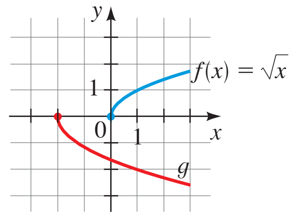

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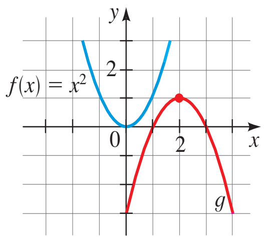
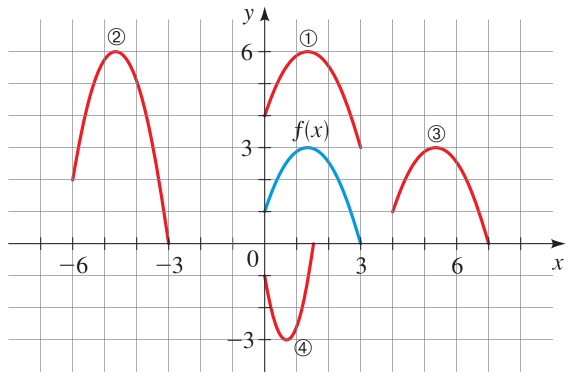


69–70 ■ Identifying Transformations The graph of $y = f(x)$ is given. Match each equation with its graph.

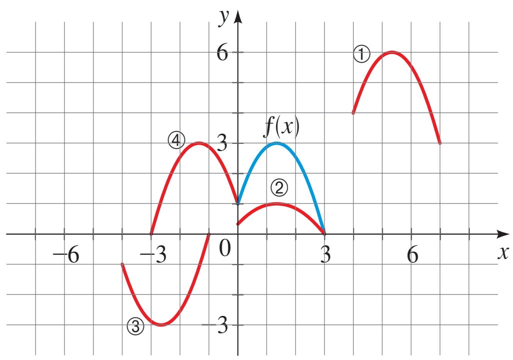
69. **(a)** $y = f(x - 4)$ **(b)** $y = f(x) + 3$

(c) $y = 2f(x + 6)$ **(d)** $y = -f(2x)$



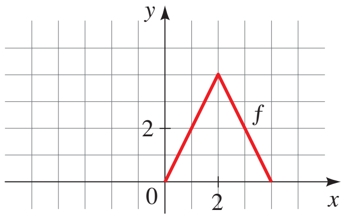
70. **(a)** $y = \frac{1}{3}f(x)$ **(b)** $y = -f(x + 4)$

(c) $y = f(x - 4) + 3$ **(d)** $y = f(-x)$

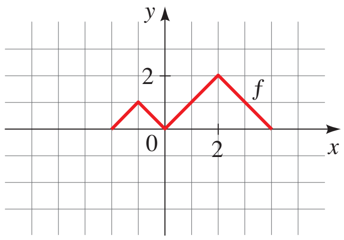


71–74 ■ Graphing Transformations The graph of a function f is given. Sketch the graphs of the following transformations of f.

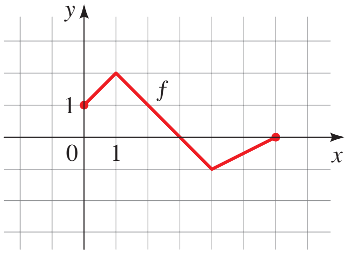
71. **(a)** $y = f(x - 2)$ **(b)** $y = f(x) - 2$
(c) $y = 2f(x)$ **(d)** $y = -f(x) + 3$
(e) $y = f(-x)$ **(f)** $y = \frac{1}{2}f(x - 1)$



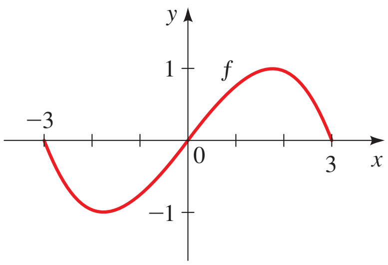
72. **(a)** $y = f(x + 1)$ **(b)** $y = f(-x)$
(c) $y = f(x - 2)$ **(d)** $y = f(x) - 2$
(e) $y = -f(x)$ **(f)** $y = 2f(x)$



73. **(a)** $y = f(2x)$ **(b)** $y = f\left(\frac{1}{2}x\right)$



74. **(a)** $y = f(3x)$ **(b)** $y = f\left(\frac{1}{3}x\right)$



75–76 ■ Graphing Transformations Use the graph of $f(x) = [\![x]\!]$ described on page 163 to graph the indicated function.

75. $y = [\![2x]\!]$ **76.** $y = [\![\frac{1}{4}x]\!]$

77–80 ■ Graphing Transformations Graph the functions on the same screen using the given viewing rectangle. How is each graph related to the graph in part (a)?

77. Viewing rectangle $[-8, 8]$ by $[-2, 8]$
(a) $y = \sqrt[4]{x}$ **(b)** $y = \sqrt[4]{x + 5}$
(c) $y = 2\sqrt[4]{x + 5}$ **(d)** $y = 4 + 2\sqrt[4]{x + 5}$

78. Viewing rectangle $[-8, 8]$ by $[-6, 6]$
(a) $y = |x|$ **(b)** $y = -|x|$
(c) $y = -3|x|$ **(d)** $y = -3|x - 5|$

79. Viewing rectangle $[-4, 6]$ by $[-4, 4]$
(a) $y = x^6$ **(b)** $y = \frac{1}{3}x^6$
(c) $y = -\frac{1}{3}x^6$ **(d)** $y = -\frac{1}{3}(x - 4)^6$

80. Viewing rectangle $[-6, 6]$ by $[-4, 4]$
(a) $y = \dfrac{1}{\sqrt{x}}$ **(b)** $y = \dfrac{1}{\sqrt{x + 3}}$
(c) $y = \dfrac{1}{2\sqrt{x + 3}}$ **(d)** $y = \dfrac{1}{2\sqrt{x + 3}} - 3$

81–82 ■ Graphing Transformations If $f(x) = \sqrt{2x - x^2}$, graph the following functions in the viewing rectangle $[-5, 5]$ by $[-4, 4]$. How is each graph related to the graph in part (a)?

81. **(a)** $y = f(x)$ **(b)** $y = f(2x)$ **(c)** $y = f\left(\frac{1}{2}x\right)$

82. **(a)** $y = f(x)$ **(b)** $y = f(-x)$
(c) $y = -f(-x)$ **(d)** $y = f(-2x)$
(e) $y = f\left(-\frac{1}{2}x\right)$

83–90 ■ Even and Odd Functions Determine whether the function f is even, odd, or neither. If f is even or odd, use symmetry to sketch its graph.

83. $f(x) = x^4$ **84.** $f(x) = x^3$

85. $f(x) = x^2 + x$ **86.** $f(x) = x^4 - 4x^2$

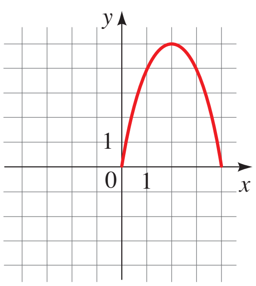
87. $f(x) = x^3 - x$ **88.** $f(x) = 3x^3 + 2x^2 + 1$

89. $f(x) = 1 - \sqrt[3]{x}$ **90.** $f(x) = x + \dfrac{1}{x}$

SKILLS Plus

91–92 ■ Graphing Even and Odd Functions The graph of a function defined for $x \geq 0$ is given. Complete the graph for $x < 0$ to make **(a)** an even function and **(b)** an odd function.

91.



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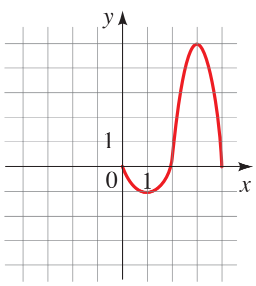
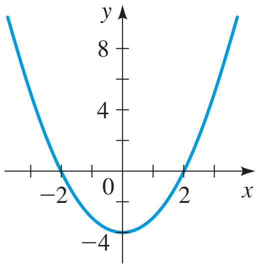
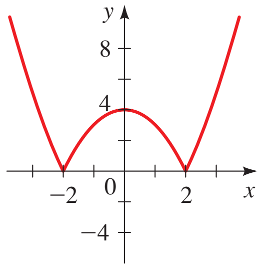


93–94 ■ Graphing the Absolute Value of a Function These exercises show how the graph of $y = |f(x)|$ is obtained from the graph of $y = f(x)$.

93. The graphs of $f(x) = x^2 - 4$ and $g(x) = |x^2 - 4|$ are shown. Explain how the graph of g is obtained from the graph of f.

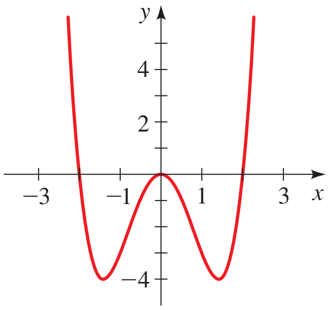


$f(x) = x^2 - 4$



$g(x) = |x^2 - 4|$

94. The graph of $f(x) = x^4 - 4x^2$ is shown. Use this graph to sketch the graph of $g(x) = |x^4 - 4x^2|$.



95–96 ■ Graphing the Absolute Value of a Function Sketch the graph of each function.

95. **(a)** $f(x) = 4x - x^2$ **(b)** $g(x) = |4x - x^2|$

96. **(a)** $f(x) = x^3$ **(b)** $g(x) = |x^3|$

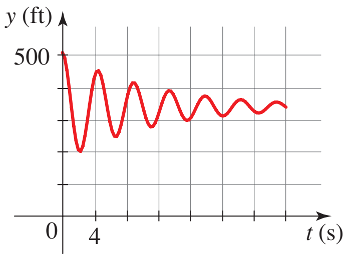
APPLICATIONS

97. Bungee Jumping Luisa goes bungee jumping from a 500-ft-high bridge. The graph shows Luisa's height $h(t)$ (in ft) after t seconds.

(a) Describe in words what the graph indicates about Luisa's bungee jump.

(b) Suppose Luisa goes bungee jumping from a 400-ft-high bridge. Sketch a new graph that shows Luisa's height $H(t)$ after t seconds.

(c) What transformation must be performed on the function h to obtain the function H? Express the function H in terms of h.

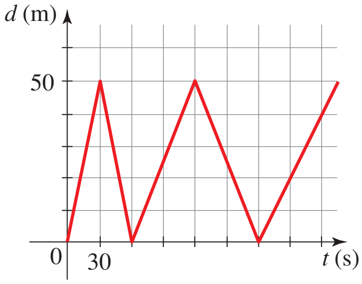


98. Swimming Laps Miyuki practices swimming laps with her team. The function $y = f(t)$ graphed below gives her distance (in meters) from the starting edge of the pool t seconds after she starts her laps.

(a) Describe in words Miyuki's swim practice. What is her average speed for the first 30 s?

(b) Graph the function $y = 1.2f(t)$. How is the graph of the new function related to the graph of the original function?

(c) What is Miyuki's new average speed for the first 30 s?

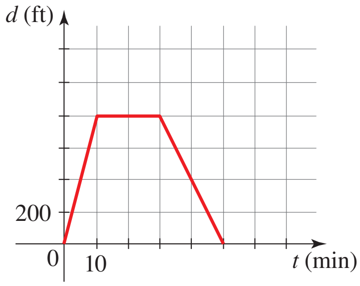


99. Field Trip A class of fourth graders walks to a park on a field trip. The function $y = f(t)$ graphed below gives their distance from school (in ft) t minutes after they left school.

(a) What is the average speed going to the park? How long was the class at the park? How far away is the park?

(b) Graph the function $y = 0.5f(t)$. How is the graph of the new function related to the graph of the original function? What is the average speed going to the new park? How far away is the new park?

(c) Graph the function $y = f(t - 10)$. How is the graph of the new function related to the graph of the original function? How does the field trip descibed by this function differ from the original trip?



DISCUSS ■ DISCOVER ■ PROVE ■ WRITE

100–101 ■ DISCUSS: Obtaining Transformations Can the function g be obtained from f by transformations? If so, describe the transformations needed.

100. The functions f and g are described algebraically as follows:

$$f(x) = (x + 2)^2 \qquad g(x) = (x - 2)^2 + 5$$