Exam 2 Practice Questions

1. Consider the iterative equation
$$x_{n+1} = 5x_n^3.$$
   (a) Compute x_1, x_2, and x_3 in terms of x_0.
   (b) Give an exact solution for x_n.
   (c) Use your answer to part (b) to find the basin of attraction for the stable fixed point 0. That is, the largest open interval I containing 0 such that if x_0 is in I then $\lim_{n\to\infty} x_n = 0$.

(a) $x_1 = 5x_0^3$

$x_2 = 5x_1^3 = 5(5x_0^3)^3 = 5(5^3 x_0^{3^2}) = 5^{1+3} x_0^{3^2}$

$x_3 = 5x_2^3 = 5(5^{1+3} x_0^{3^2})^3 = 5(5^{3+3^2} x_0^{3^3}) = 5^{1+3+3^2} x_0^{3^3}$

(b) IN GENERAL, WE HAVE

$$x_n = 5^{1+3+3^2+\dots+3^{n-1}} x_0^{3^n} = 5^{\sum_{i=0}^{n-1} 3^i} x_0^{3^n}$$

$$= 5^{\frac{3^n-1}{3-1}} x_0^{3^n} = (5^{\frac{1}{2}})^{3^n-1} x_0^{3^n}$$

$$\boxed{x_n = \frac{1}{\sqrt{5}}\left(\sqrt{5}\,x_0\right)^{3^n}}$$

(c) RECALL THAT $\lim_{n\to\infty} a^n = \begin{cases} 0 & \text{IF } |a| < 1 \\ 1 & \text{IF } a = 1 \\ \text{D.N.E.} & \text{OTHERWISE} \end{cases}$

$$\therefore \lim_{n\to\infty} x_n = \frac{1}{\sqrt{5}} \lim_{n\to\infty} \left(\sqrt{5}\,x_0\right)^{3^n} = 0$$

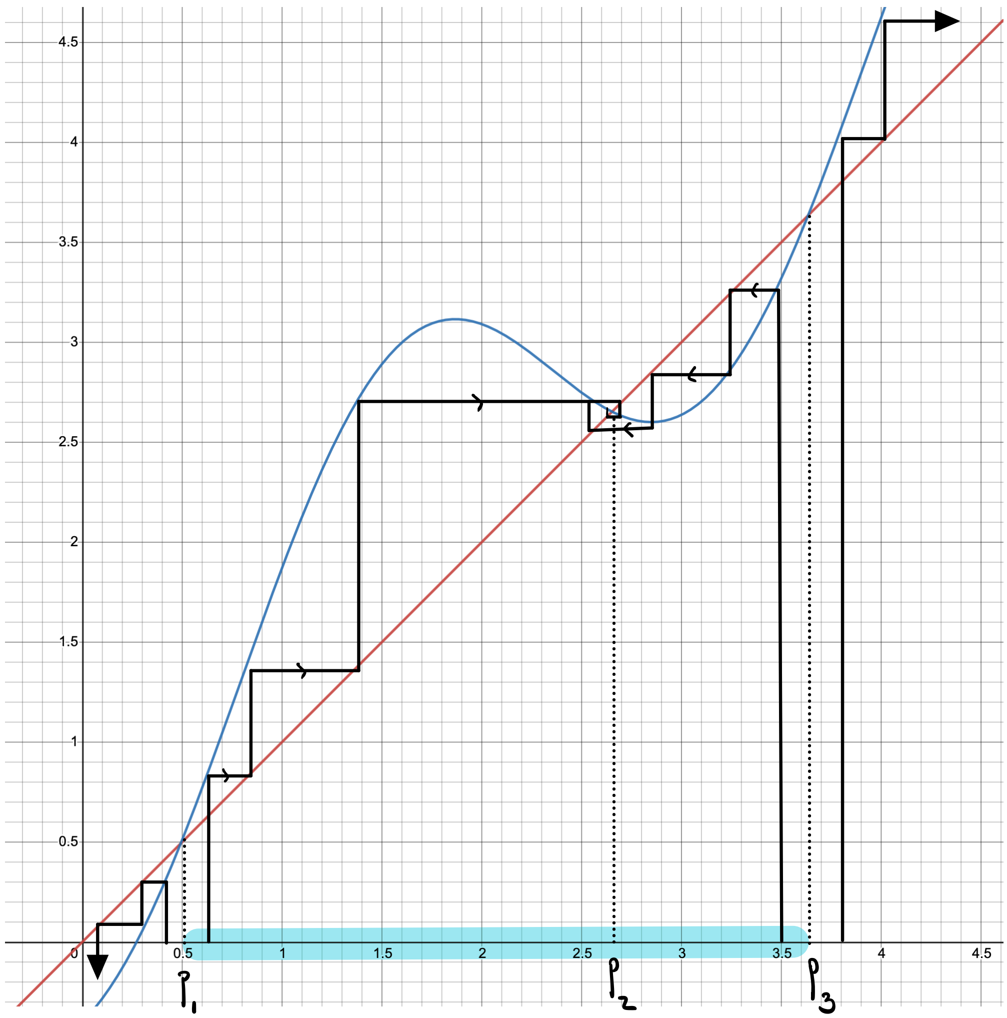
IF & ONLY IF

$$-1 < \sqrt{5}\,x_0 < 1$$

$$\boxed{-\frac{1}{\sqrt{5}} < x_0 < \frac{1}{\sqrt{5}}}$$

jadamski1@fordham.edu

2. The graphs of $y = f(x)$ and $y = x$ are shown below. Suppose the three intersections of the two graphs are (p_1, p_1), (p_2, p_2), and (p_3, p_3), with $p_1 < p_2 < p_3$.
   (a) Use cobwebbing to identify all (visible) fixed points and classify each one as either stable or unstable.
   (b) Use cobwebbing to determine the basin of attraction of each (if any) stable fixed point.
   (c) Which, if any, of the fixed points exhibit oscillation of nearby solutions?



(a) UNSTABLE STABLE, OSCILLATION UNSTABLE

(b) BASIN OF ATTRACTION FOR p_2 IS (p_1, p_3)

(c) ONLY p_2.

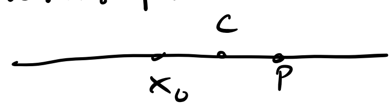
3. Let $\epsilon > 0$. If p is a fixed point of the iterative equation $x_{n+1} = f(x_n)$, and $0 < f'(x) \leq 1/2$ for all x such that $|x - p| < \epsilon$, show that p is locally stable. That is, show that

$$\text{if } |x_0 - p| < \epsilon \text{ then } \lim_{n\to\infty} |x_n - p| = 0.$$

Note: I am asking you to provide a part of the proof of the Stability and Oscillation Theorem. Thus, you cannot use the theorem to prove the result. That is, you cannot simply say that the fixed point p is stable because $|f'(p)| < 1$.

MEAN VALUE THEOREM STATES THAT $f(x_0) - f(p) = f'(c)(x_0 - p)$

FOR SOME $\underbrace{c \text{ BETWEEN } x_0 \text{ \& } p}$.



$\Rightarrow |c-p| < |x_0 - p| < \varepsilon \Rightarrow 0 < f'(c) \leq \frac{1}{2}$.

$\therefore \ |f(x_0) - f(p)| \leq \frac{1}{2}|x_0 - p|$, i.e. $|x_1 - p| \leq \frac{1}{2}|x_0 - p|$.

NOTE $|x_1 - p| \leq \frac{1}{2}|x_0 - p| < |x_0 - p| < \varepsilon$,

THUS THE SAME ARGUMENT SHOWS $|x_2 - p| \leq \frac{1}{2}|x_1 - p| \leq \left(\frac{1}{2}\right)^2 |x_0 - p|$

AND, MORE GENERALLY, $|x_n - p| \leq \left(\frac{1}{2}\right)^n |x_0 - p|$.

THUS

$$0 \leq \lim_{n\to\infty} |x_n - p| \leq \lim_{n\to\infty} \left(\frac{1}{2}\right)^n |x_0 - p| = 0$$

AND SO $\lim_{n\to\infty} |x_n - p| = 0$. ■

4. Let p be the unique solution to the equation

$$e^{2x} = 4 - x^3.$$

Assuming x_0 is sufficiently close to p, use Newton's method of root-finding to give an iterative equation $x_{n+1} = f(x_n)$ such that x_n converges to p.

LET $g(x) = e^{2x} + x^3 - 4$. THEN WE SEEK A ROOT OF g.

$g'(x) = 2e^{2x} + 3x^2$ (NOTE: $g'(x) > 0$ FOR ALL x.)

NEWTON'S METHOD: $x_{n+1} = x_n - \dfrac{g(x_n)}{g'(x_n)}$ (PROVIDED $g'(x_n) \neq 0$)

$$\therefore \boxed{x_{n+1} = x_n - \frac{e^{2x_n} + x_n^3 - 4}{2e^{2x_n} + 3x_n^2}}$$

5. For

$$f(x) = 1 - 2\left|x - \frac{1}{2}\right|,$$

$p_1 = 2/33$ is one point of an m-cycle.

(a) Find all other points of that cycle, and state the period of p.

(b) Determine the stability of that cycle and justify your answer.

and stability.

$p_2 = f(p_1) = 2\left(\frac{2}{33}\right) = 4/33$

$p_3 = f(p_2) = 2(4/33) = 8/33$

$p_4 = f(p_3) = 2(8/33) = 16/33$

$p_5 = f(p_4) = 2(16/33) = 32/33$

$p_6 = f(p_5) = 2 - 2(32/33) = 66/33 - 64/33 = 2/33 = p_1$

NOTE: $f(x) = \begin{cases} 2x & \text{IF } x \leq 1/2 \\ 2 - 2x & \text{IF } x > 1/2 \end{cases}$

$f'(x) = \begin{cases} 2 & \text{IF } x < 1/2 \\ \text{UND.} & \text{IF } x = 1/2 \\ -2 & \text{IF } x > 1/2 \end{cases}$

$\therefore$ 2/33 IS A POINT OF $\boxed{\text{PERIOD } 5.}$

SINCE $|f'(p_1) f'(p_2) \cdots f'(p_5)| = 2^5 > 1$,

THE 5-CYCLE IS $\boxed{\text{UNSTABLE.}}$

6. Consider the one-parameter family of functions

$$f_r(x) = rx(3 - x^2),$$

with $r > 0$.

(a) Find the interval of stability for the fixed point 0.

(b) Find the positive fixed point p_r and its interval of existence.

(c) Find the interval of stability for p_r.

(a) $f_r(x) = 3rx - rx^3$

$f_r'(x) = 3r - 3rx^2 \Rightarrow f_r'(0) = 3r$

0 IS STABLE FOR ALL $r > 0$ SUCH THAT $|f_r'(0)| < 1$

$\Rightarrow 3r < 1 \Rightarrow r < \frac{1}{3}$. $\therefore$ $\boxed{0 < r < \frac{1}{3}}$

(b) $f_r(x) = 3rx - rx^3 = x \Rightarrow rx^3 + x - 3rx = 0$

$x(rx^2 + 1 - 3r) = 0 \Rightarrow x = 0$ (NOT POSITIVE) OR $rx^2 + 1 - 3r = 0$

$x^2 = \frac{3r-1}{r} \Rightarrow x = \pm\sqrt{\frac{3r-1}{r}}$ $(r > 0)$

THUS, A POSITIVE FIXED POINT $p_r = \sqrt{\frac{3r-1}{r}}$

EXISTS WHEN $3r - 1 \geq 0 \Rightarrow r \geq \frac{1}{3}$

(c) $f_r'(x) = 3r - 3rx^2 \Rightarrow f_r'(p_r) = 3r - 3r\left(\frac{3r-1}{r}\right) = 3r - 9r + 3$

p_r IS STABLE FOR $r \geq \frac{1}{3}$ SUCH THAT

$|f_r'(p_r)| = |3 - 6r| < 1$

$\Rightarrow -1 < 3 - 6r < 1 \Rightarrow -4 < -6r < -2$

$\Rightarrow 2 < 6r < 4 \Rightarrow$ $\boxed{\frac{1}{3} < r < \frac{2}{3}}$

9. Use the graphs of f and its iterates f^2 and f^3 below to answer the following questions. Explain your answers.
   (a) How many fixed points does f have? 2
   (b) How many 2-cycles does f have? 1 (2 POINTS OF PERIOD 2)
   (c) How many 3-cycles does f have? 2 (6 POINTS OF PERIOD 3)
   (d) Let p be the largest fixed point. Do solutions oscillate locally around p? YES, $f'(p) < 0$