Homework 5

Sections 3.4-6

Due Monday 11/7

1. (8 points) Let p be the unique solution of the equation

$$\sin^2 x = x + \frac{1}{x^2}.$$

Assuming x_0 is sufficiently close to p, use Newton's method of root-finding to give an iterative equation such that x_n converges to p.

SOLUTIONS TO THE GIVEN EQUATION ARE ROOTS OF

$$g(x) = x + \frac{1}{x^2} - \sin^2 x.$$

NEWTON'S METHOD SHOWS THAT ROOTS OF g ARE STABLE FIXED POINTS OF $f(x) = x - \frac{g(x)}{g'(x)}$.

NOTE: $g'(x) = 1 - \frac{2}{x^3} - 2\sin x \cos x$

$\therefore$ ITERATIVE EQUATION IS $x_{n+1} = f(x_n)$,

i.e.

$$\boxed{x_{n+1} = x_n - \frac{x_n + \frac{1}{x_n^2} - \sin^2 x_n}{1 - \frac{2}{x_n^3} - 2\sin x_n \cos x_n}}$$

2. (10 points) Use Newton's method of root-finding to find increasingly better apoproximate solutions x_1, x_2, x_3, and x_4 to the equation

$$g(x) = x^2 - 7 = 0,$$

starting with initial guess $x_0 = 3$. What is $\lim_{n\to\infty} x_n$?

NEWTON'S METHOD : $x_{n+1} = x_n - \dfrac{g(x_n)}{g'(x_n)}$

HERE, $g(x) = x^2 - 7$, $g'(x) = 2x$

$\therefore \quad x_{n+1} = x_n - \dfrac{x_n^2 - 7}{2x_n}$

$\lim_{n\to\infty} x_n$ IS THE POSITIVE ROOT OF g, $\boxed{\sqrt{7}}$

1700 HW 5

	A	B	C
1	Newtown's Method for g(x) = x^2 - 7		
2	n	x_n	
3	0	3	
4	1	2.666666667	
5	2	2.645833333	
6	3	2.645751312	
7	4	2.645751311	

3. (8 points) Use Newton's method of root-finding to give an iterative equation $x_{n+1} = f(x_n)$ and an initial value x_0 such that x_n converges to $a^{1/p}$, where a and p are both integers ≥ 2.
Hint: Let $x = a^{1/p}$. Then we are trying to find a solution to $x^p = a$. To find the initial value x_0, you could use a cobwebbing diagram to visualize the basin of attraction of the fixed point of Newton's equation $x_{n+1} = f(x_n)$.

LET $x = a^{1/p}$. WE MUST APPROXIMATE x.

$x^p = a$

$x^p - a = 0$.

SET $g(x) = x^p - a$. WE MUST FIND A (POSITIVE, REAL) ROOT OF g.

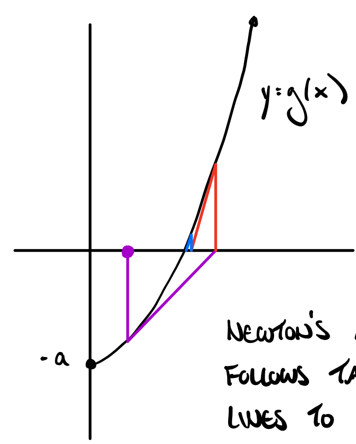
NEWTON'S METHOD: $x_{n+1} = x_n - \dfrac{g(x_n)}{g'(x_n)}$

$$\boxed{x_{n+1} = \underbrace{x_n - \frac{x_n^p - a}{p\, x_n^{p-1}}}_{\text{CALL THIS } f(x_n)}}$$

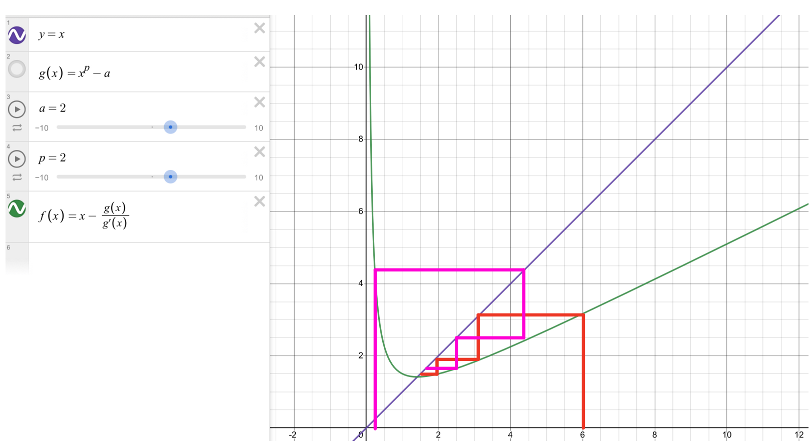
IF WE CONSIDER THE GRAPH $y = g(x) = x^p - a$, WHERE a & p ARE BOTH INTEGERS ≥ 2,

WE SEE THAT $x_n \to a^{1/p}$ FOR ALL $x_0 > 0$.
FOR EXAMPLE, $x_0 = 1$ OR $x_0 = a$.

(NO JUSTIFICATION IS NEEDED FOR CHOICE OF x_0)



NEWTON'S METHOD FOLLOWS TANGENT LINES TO FIND BETTER APPROXIMATIONS TO A ROOT OF g.



ALTERNATIVELY, COBWEBBING SHOWS BASIN OF ATTRACTION OF FIXED POINT OF f IS $(0, \infty)$.

4. Find all fixed points of f and then use them to help find all points of period 2.

   (a) (8 points) $f(x) = -x^2 + 2x + 2$

FIXED POINTS: $f(x) = -x^2 + 2x + 2 = x$

$$x^2 - x - 2 = 0$$

$$(x-2)(x+1) = 0 \quad \Rightarrow \quad x = -1, 2$$

POINTS OF PERIOD 2: $f^2(x) = -(-x^2+2x+2)^2 + 2(-x^2+2x+2)^2 + 2 = x$

NOTE THAT FIXED POINTS OF f ARE SOLUTIONS TO THIS EQ, i.e. ROOTS OF THIS POLYNOMIAL.

$$\begin{array}{r|l} & x^2 - 3x + 1 \\ \hline x^2 - x - 2 & x^4 - 4x^3 + 2x^2 + 5x - 2 \\ & -(x^4 - x^3 - 2x^2) \\ \hline & -3x^3 + 4x^2 + 5x - 2 \\ & -(-3x^3 + 3x^2 + 6x) \\ \hline & x^2 - x - 2 \\ & -(x^2 - x - 2) \\ \hline & 0 \end{array}$$

$$x^4 - 4x^3 + 2x^2 + 5x - 2 = 0$$

$$(x^2 - x - 2)(x^2 - 3x + 1) = 0$$

$$x = \underbrace{-1, 2}_{\text{FIXED POINTS}}, \underbrace{\boxed{\frac{3}{2} \pm \frac{\sqrt{5}}{2}}}_{\text{POINTS OF PERIOD 2}}$$

(b) (8 points) $f(x) = \dfrac{2}{x} - x$

FIXED POINTS OF f: $f(p) = p$

$$\frac{2}{p} - p = p \implies 2 - p^2 = p^2 \implies p^2 = 1$$

$$p = \pm 1.$$

FIXED POINTS OF f^2: $f(f(x)) = x$

$$\frac{2}{\frac{2}{x} - x} - \left(\frac{2}{x} - x\right) = x$$

$$\frac{2x}{2 - x^2} - \frac{2}{x} = 0$$

$$2x^2 - 2(2 - x^2) = 0$$

$$2x^2 - 4 + 2x^2 = 0$$

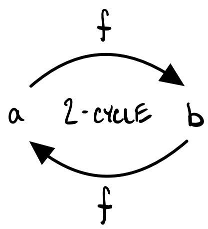
$$4x^2 - 4 = 0$$

$$4(x+1)(x-1) = 0$$

$x = \pm 1 \longleftarrow$ BUT THESE ARE FIXED POINTS!

$\therefore$ $\boxed{\text{THERE ARE NO POINTS OF PERIOD 2 \& NO 2-CYCLES.}}$

5. (8 points) Given that $a = 1$ is one point of a 2-cycle of $f(x) = 2 - 2^x$, find the other point b and determine the stability of the cycle.



$b = f(a) = f(1) = 2 - 2^1 = 0$

$a = f(b) = f(0) = 2 - 2^0 = 1$ ✓

$f'(x) = 2^x \ln 2$, $f'(1) = 2\ln 2$
$f'(0) = \ln 2$

$\left| f'(0) f'(1) \right| = \left| 2(\ln 2)^2 \right| \approx .9609 < 1$

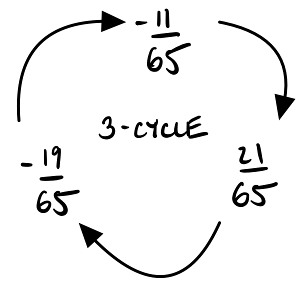
$\Rightarrow$ $\boxed{\{0, 1\} \text{ IS A STABLE 2-CYCLE}}$

6. (8 points) For $f(x) = 1 - 4|x|$, $p_1 = -11/65$ is a point of an m-cycle. Find all other points of that cycle, and determine its period and stability.

$$p_2 = f(p_1) = 1 - 4\left|-{}^{11}/_{65}\right| = {}^{21}/_{65}$$

$$p_3 = f(p_2) = 1 - 4\left|{}^{21}/_{65}\right| = -{}^{19}/_{65}$$

$$p_4 = f(p_3) = 1 - 4\left|-{}^{19}/_{65}\right| = -{}^{11}/_{65} = p_1$$



$$f'(x) = \begin{cases} -4 & \text{IF } x > 0 \\ \text{UND.} & \text{IF } x = 0 \\ 4 & \text{IF } x < 0 \end{cases}$$

$$\left|f'(p_1)f'(p_2)f'(p_3)\right| = 4^3 > 1 \implies \boxed{\left\{-\frac{11}{65}, \frac{21}{65}, -\frac{19}{65}\right\} \text{ IS AN UNSTABLE 3-CYCLE.}}$$

7. (a) (6 points) Suppose $f^4(p) = p$ for some point p and some function $f(x)$. What are the possible periods that p might have?

SINCE THE PERIOD OF p IS (BY DEFINITION) THE SMALLEST POSITIVE INTEGER m SUCH THAT $f^m(p) = p$, WE HAVE $m \leq 4$.

BUT $m = 3$ IS IMPOSSIBLE:

IF $f^3(p) = p$ THEN $p = f^4(p) = f(f^3(p)) = f(p)$.

$\therefore$ $f(p) = p$ & SO p IS A FIXED POINT.

$\therefore$ $\boxed{\text{POSSIBLE PERIODS OF } p \text{ ARE } 1, 2, 4.}$

CORRECT ANSWER WITHOUT PROOF IS OK.

(b) (6 points) Suppose $f^{12}(p) = p$ for some point p and some function $f(x)$. What are the possible periods that p might have?

NOW THE PERIOD m OF THE POINT p IS NO GREATER THAN 12.

SUPPOSE $m < 12$. WRITE $12 = qm + r$ (q QUOTIENT, r REMAINDER) WITH $0 \leq r < m$

THEN $p = f^{12}(p) = f^r(f^{qm}(p)) = f^r(p)$

IF $r > 0$ THEN THIS SHOWS THE PERIOD OF p IS NO GREATER THAN r.

THAT IS, $m \leq r$.

BUT $r < m$, CONTRADICTION $\Rightarrow\!\Leftarrow$

THUS $r = 0$ & THE PERIOD m IS A DIVISOR OF 12.

THAT IS, $\boxed{\text{POSSIBLE PERIODS OF } p \text{ ARE } 1, 2, 3, 4, 6, 12.}$

CORRECT ANSWER WITHOUT PROOF IS OK.

8. (10 points) Find the interval of stability of the fixed point 0 for $r > 0$ of $f_r(x) = rx^2(1-x)$.

THE FIXED POINT 0 IS STABLE FOR ALL VALUES OF $r > 0$ SUCH THAT $|f_r'(0)| < 1$.

$f_r'(x) = 2rx - 3rx^2 \Rightarrow |f_r'(0)| = 0 < 1$ ALWAYS.

$\therefore$ $\boxed{\text{INTERVAL OF STABILITY FOR } 0 \text{ IS } r > 0}$

9. (10 points) Find all positive fixed points of $f_c(x) = \dfrac{cx^2}{x^2+1}$ and their intervals of existence for $c > 0$.

FIXED POINTS : SOLVE $\dfrac{cx^2}{x^2+1} = x \implies cx^2 = x^3 + x$

$0 = x^3 - cx^2 + x = x(x^2 - cx + 1)$

$\implies x = 0$ or $x = \dfrac{c \pm \sqrt{c^2-4}}{2}$

NOT POSITIVE

ONLY DEFINED IF $c^2 \geq 4 \iff |c| \geq 2$.

ONLY POSITIVE IF $c \geq 2$.

THE POSITIVE FIXED POINT $\dfrac{c \pm \sqrt{c^2-4}}{2}$

EXISTS FOR ALL $c \geq 2$.

10. (10 points) Find all 2-cycles of $f_a(x) = -ax^3$ and their intervals of existence and stability for $a > 0$.

FIXED POINTS OF f : $f_a(x) = -ax^3 = x \Rightarrow 0 = x\underbrace{(1 + ax^2)}_{\neq 0 \text{ WHEN } a > 0.} \Rightarrow x = 0$

FIXED POINTS OF f^2 : $f_a^2(x) = -a(-ax^3)^3 = x \Rightarrow a^4x^9 = x$

$$0 = x(1 - a^4x^8) \Rightarrow x = 0 \text{ (FIXED POINT OF } f\text{)}$$

$$x^8 = a^{-4} \Rightarrow x = \pm a^{-1/2}, \text{ i.e. } x = \pm \frac{1}{\sqrt{a}}$$

$\therefore$ $\boxed{\text{2 CYCLE } \left\{ \frac{1}{\sqrt{a}}, -\frac{1}{\sqrt{a}} \right\} \text{ EXIST FOR ALL } a > 0.}$

STABILITY: $f_a'(x) = -3ax^2$

$$\left| f_a'\left(\frac{1}{\sqrt{a}}\right) f'\left(-\frac{1}{\sqrt{a}}\right) \right| = \left| \left(-3a\left(\frac{1}{\sqrt{a}}\right)^2\right)\left(-3a\left(-\frac{1}{\sqrt{a}}\right)^2\right) \right| = |(-3)(-3)| = 9 > 1$$

$\therefore$ $\boxed{\text{2 CYCLE } \left\{ \frac{1}{\sqrt{a}}, -\frac{1}{\sqrt{a}} \right\} \text{ IS UNSTABLE FOR ALL } a \text{ (EMPTY INTERVAL OF STABILITY)}}$

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