Exam 2

1. Consider the non-linear model
$$x_{n+1} = (x_n - 8)^3 + 8.$$

   (a) (8 points) Give an exact solution for x_n.
   Hint: You might want to first find an exact solution for $u_n = x_n - 8$.

   (b) (8 points) Use the exact solution from part (a) to determine all possible values of $\lim_{n\to\infty} x_n$ and to find the basin of attraction of any stable fixed point(s).

(a) $x_{n+1} - 8 = (x_n - 8)^3$

LET $u_n = x_n - 8$. THEN $u_{n+1} = u_n^3$.

$\Rightarrow$ $u_n = u_0^{3^n}$

$x_n - 8 = (x_0 - 8)^{3^n}$ $\Rightarrow$ $\boxed{x_n = (x_0 - 8)^{3^n} + 8}$

(b) RECALL:
$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{IF } |r| < 1 \\ 1 & \text{IF } r = 1 \\ \text{D.N.E.} & \text{ELSE} \end{cases}$$

STABLE $\downarrow$ BASIN OF ATTRACTION $\downarrow$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} (x_0 - 8)^{3^n} + 8 = \begin{cases} 0 + 8 = 8 & \text{IF } |x_0 - 8| < 1, \ x_0 \in (7, 9) \\ 1 + 8 = 9 & \text{IF } x_0 - 8 = 1, \ x_0 = 9 \\ -1 + 8 = 7 & \text{IF } x_0 - 8 = -1, \ x_0 = 7 \\ \text{D.N.E.} & \text{ELSE.} \end{cases}$$

2. (12 points) Given that $p = 1$ is a fixed point of the equation
$$x_{n+1} = \frac{1}{3}(x_n^2 - 3x_n + 5),$$
use the derivative to determine the stability of $p = 1$ and to state whether solutions that begin near $p = 1$ oscillate or not.

$f(x) = \frac{1}{3}(x^2 - 3x + 5)$

$f'(x) = \frac{1}{3}(2x - 3)$

$f'(p) = f'(1) = -\frac{1}{3}$

SINCE $|f'(p)| = |-\frac{1}{3}| < 1$, p IS STABLE.

SINCE $f'(p) = -\frac{1}{3} < 0$, SOLUTIONS OSCILLATE NEAR p.

3. (12 points) Let p be the unique solution to the equation

$$\sin x = \ln x.$$

Assuming x_0 is sufficiently close to p, use Newton's method of root-finding to give an iterative equation $x_{n+1} = f(x_n)$ such that x_n converges to p.

LET $g(x) = \sin x - \ln x$

$g'(x) = \cos x - \frac{1}{x}$

NEWTON'S METHOD : $x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}$

$$\boxed{x_{n+1} = x_n - \frac{\sin x_n - \ln x_n}{\cos x_n - \frac{1}{x_n}}}$$

4. (16 points) Find the 2-cycle of

$$f(x) = x^2 - 2x$$

and determine if it is stable or unstable.

FIXED POINTS: $f(x) = x$: $x^2 - 2x = x$

$x^2 - 3x = 0$

$x(x-3) = 0 \Rightarrow x = 0, 3$

FIXED POINTS OF $f^2(x)$: $f(f(x)) = x$

$(x^2-2x)^2 - 2(x^2-2x) = x$

$x^4 - 4x^3 + 4x^2 - 2x^2 + 4x = x$

$x^4 - 4x^3 + 2x^2 + 3x = 0$

$$\begin{array}{r|l} & x^2 - x - 1 \\ \hline x^2-3x & x^4 - 4x^3 + 2x^2 + 3x \\ & -(x^4 - 3x^3) \\ \hline & -x^3 + 2x^2 + 3x \\ & -(-x^3 + 3x^2) \\ \hline & -x^2 + 3x \\ & -(-x^2 + 3x) \\ \hline & 0 \end{array}$$

WE KNOW $x = 0$ & $x = 3$ ARE SOLUTIONS

$\Rightarrow x(x-3) = x^2 - 3x$ IS A FACTOR!

$x(x-3)\underbrace{(x^2 - x - 1)} = 0$

POINTS OF PERIOD 2: $x^2 - x - 1 = 0 \Rightarrow \boxed{x = \frac{1 \pm \sqrt{5}}{2}}$

$f'(x) = 2x - 2$.

$\left| f'\left(\frac{1+\sqrt{5}}{2}\right) f'\left(\frac{1-\sqrt{5}}{2}\right) \right| = \left| (-1+\sqrt{5})(-1-\sqrt{5}) \right| = \left| (-1)^2 - \sqrt{5}^2 \right|$

$= |1 - 5| = 4 > 1$

$\therefore$ THE 2-CYCLE IS $\boxed{\text{UNSTABLE}}$.

5. (12 points) Suppose there are

- 2 solutions to $f(x) = x$,
- 8 solutions to $f^2(x) = x$,
- 8 solutions to $f^3(x) = x$, and
- 20 solutions to $f^6(x) = x$.

How many 2-cyles, 3-cycles, and 6-cycles does f have?

8 SOLUTIONS TO $f^2(x) = x$ INCLUDE 2 FIXED POINTS.

$\Rightarrow$ 6 POINTS OF PERIOD 2 $\Rightarrow$ $\boxed{\text{3 2-CYCLES}}$

8 SOLUTIONS TO $f^3(x) = x$ INCLUDE 2 FIXED POINTS.

$\Rightarrow$ 6 POINTS OF PERIOD 3 $\Rightarrow$ $\boxed{\text{2 3-CYCLES}}$

20 SOLUTIONS TO $f^6(x) = x$ INCLUDE 2 FIXED POINTS, 6 POINTS OF PERIOD 2, AND 6 POINTS OF PERIOD 3.

$\Rightarrow$ 6 POINTS OF PERIOD 6 $\Rightarrow$ $\boxed{\text{1 6-CYCLE}}$

6. (16 points) Consider the parametrized family

$$f_r(x) = x\sqrt{r - x}, \quad r > 0.$$

Find the positive fixed point(s) and the interval of existence and interval of stability for each.

$f(x) = x \quad : \quad x\sqrt{r-x} = x$

$$x\sqrt{r-x} - x = x\left(\sqrt{r-x} - 1\right) = 0$$

$$x = 0, \quad \sqrt{r-x} = 1$$

$$r - x = 1 \quad \Rightarrow \quad x = r - 1$$

$\therefore$ $\boxed{\text{$r-1$ IS A POSITIVE FIXED POINT WHEN $r > 1$}}$

$$f'(x) = \sqrt{r-x} - x\frac{1}{2\sqrt{r-x}}$$

$$\left|f'(r-1)\right| = \left|\sqrt{r-(r-1)} - (r-1)\frac{1}{2\sqrt{r-(r-1)}}\right| = \left|1 - \tfrac{1}{2}(r-1)\right| = \left|\tfrac{3}{2} - \tfrac{r}{2}\right| < 1$$

$$-1 < \frac{3}{2} - \frac{r}{2} < 1 \Rightarrow -2 < 3 - r < 2 \Rightarrow -5 < -r < -1 \Rightarrow 1 < r < 5$$

$\therefore$ $\boxed{\text{$r-1$ IS A STABLE (POSITIVE) FIXED POINT WHEN $1 < r < 5$}}$

7. In this question we will investigate the *infinite continued fraction* below.

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\cdots}}}} \tag{1}$$

Let $x_0 > 0$ and

$$x_1 = 1+\frac{1}{x_0}, \qquad x_2 = 1+\cfrac{1}{1+\frac{1}{x_0}}, \qquad x_2 = 1+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{x_0}}}, \qquad x_3 = 1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{x_0}}}}, \qquad \text{etc.}$$

(a) (4 points) Find a function f such that $x_{n+1} = f(x_n)$.

(b) (4 points) Find the positive fixed point p of f and use f' to determine whether it is stable or unstable.

(c) (4 points) The graphs $y = f(x)$ and $y = x$ are shown below. Use cobwebbing to determine the basin of attraction of the positive fixed point p.

(d) (4 points) Given $x_0 > 0$, what is the limit $\lim\limits_{n\to\infty} x_n$?
Note: This is precisely the value of the continued fraction (1).

(a) $\boxed{x_{n+1} = 1 + \frac{1}{x_n}}$, i.e. $f(x) = 1 + \frac{1}{x}$

(b) $p = 1 + \frac{1}{p} \implies p^2 - p - 1 = 0 \implies p = \frac{1 \pm \sqrt{5}}{2}$

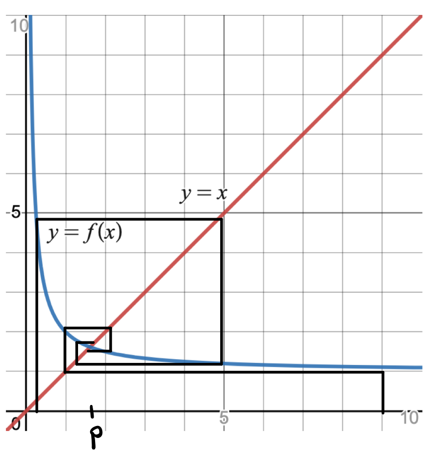
$\therefore$ Positive fixed point $\boxed{p = \frac{1+\sqrt{5}}{2}}$

$f'(x) = \frac{-1}{x^2}$

$$|f'(p)| = \left| \frac{-1}{\left(\frac{1+\sqrt{5}}{2}\right)^2} \right| = \left| \frac{-4}{(1+\sqrt{5})^2} \right| = \left| \frac{-4}{1+2\sqrt{5}+5} \right| < 1$$

$\therefore$ p is $\boxed{\text{stable.}}$

(c)



Basin of attraction appears to be $\boxed{(0,\infty)}$

(d) Since $x_0 > 0$ is in the basin of attraction,

$$\lim_{n\to\infty} x_n = p = \boxed{\frac{1+\sqrt{5}}{2}}$$

"Golden ratio" Φ