Exam 2

Answer all 7 questions for a total of 100 points. Write your solutions in the accompanying blue book, and put a box around your final answers. If you solve the problems out of order, please skip pages so that your solutions stay in order. Good luck!

1. Consider the non-linear model
$$x_{n+1} = (x_n - 8)^3 + 8.$$
   (a) (8 points) Give an exact solution for x_n.
   Hint: You might want to first find an exact solution for $u_n = x_n - 8$.
   (b) (8 points) Use the exact solution from part (a) to determine all possible values of $\lim_{n\to\infty} x_n$ and to find the basin of attraction of any stable fixed point(s).

2. (12 points) Given that $p = 1$ is a fixed point of the equation
$$x_{n+1} = \frac{1}{3}(x_n^2 - 3x_n + 5),$$
use the derivative to determine the stability of $p = 1$ and to state whether solutions that begin near $p = 1$ oscillate or not.

3. (12 points) Let p be the unique solution to the equation
$$\sin x = \ln x.$$
Assuming x_0 is sufficiently close to p, use Newton's method of root-finding to give an iterative equation $x_{n+1} = f(x_n)$ such that x_n converges to p.

4. (16 points) Find the 2-cycle of
$$f(x) = x^2 - 2x$$
and determine if it is stable or unstable.

5. (12 points) Suppose there are
   - 2 solutions to $f(x) = x$,
   - 8 solutions to $f^2(x) = x$,
   - 8 solutions to $f^3(x) = x$, and
   - 20 solutions to $f^6(x) = x$.

   How many 2-cyles, 3-cycles, and 6-cycles does f have?

6. (16 points) Consider the parametrized family
$$f_r(x) = x\sqrt{r - x}, \quad r > 0.$$
Find the positive fixed point(s) and the interval of existence and interval of stability for each.

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7. In this question we will investigate the *infinite continued fraction* below.

$$1 + \cfrac{1}{1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\cdots}}}} \tag{1}$$

Let $x_0 > 0$ and

$$x_1 = 1 + \frac{1}{x_0}, \qquad x_2 = 1 + \cfrac{1}{1 + \cfrac{1}{x_0}}, \qquad x_2 = 1 + \cfrac{1}{1 + \cfrac{1}{1+\cfrac{1}{x_0}}}, \qquad x_3 = 1 + \cfrac{1}{1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{x_0}}}}, \qquad \text{etc.}$$

(a) (4 points) Find a function f such that $x_{n+1} = f(x_n)$.

(b) (4 points) Find the positive fixed point p of f and use f' to determine whether it is stable or unstable.

(c) (4 points) The graphs $y = f(x)$ and $y = x$ are shown below. Use cobwebbing to determine the basin of attraction of the positive fixed point p.

(d) (4 points) Given $x_0 > 0$, what is the limit $\lim\limits_{n\to\infty} x_n$?

Note: This is precisely the value of the continued fraction (1).