

Exam 1

Answer all 8 questions for a total of 100 points. Write your solutions in the accompanying blue book, and put a box around your final answers. If you solve the problems out of order, please skip pages so that your solutions stay in order. Good luck!

1. (7 points) Give a linear model $x_{n+1} = a_n x_n + b_n$ such that for any initial value x_0 , the sequence

$$x_0, \quad x_1 = 1 - x_0, \quad x_2 = \frac{1}{2}x_1 - 2, \quad x_3 = 4 - \frac{1}{4}x_2, \quad x_4 = \frac{1}{8}x_3 - 8, \dots$$

is a solution.

2. (7 points) Consider the linear model

$$x_{n+1} = \frac{n+3}{n+1}x_n, \quad x_0 = 1.$$

The exact solution to this model is a product that simplifies to a relatively simple expression. Find that simplified solution for x_n .

3. Find the following sums *precisely*. Your answers *can* contain fractions and exponents. Your answers *cannot* be rounded decimal numbers computed with a calculator.

- (a) (6 points) Find the finite sum.

$$6 - \frac{6}{7} + \frac{6}{7^2} - \frac{6}{7^3} + \frac{6}{7^4} - \frac{6}{7^5} + \dots - \frac{6}{7^{15}}$$

- (b) (6 points) Find the infinite sum (if it exists).

$$\sum_{n=0}^{\infty} \frac{3^{n+2}}{5^n}$$

4. In the United States, Election day is held each year on the Tuesday after the first Monday in November. Assume 70% of eligible voters who vote (V) one year will also vote the following year, and 85% of eligible voters who do not vote (D) one year will also not vote the next year.

- (a) (2 points) Draw a transition diagram that summarizes the survey data.

- (b) (6 points) Give a linear model that describes how the proportion of eligible voters who vote V_n evolves from one year to the next.

- (c) (2 points) If 50% of eligible voters voted last year, what proportion of eligible voters will vote this year?

- (d) (6 points) What proportion of eligible voters should we expect to vote each year in the long run?

5. Uh oh! You've discovered a colony of P_0 ants living in your kitchen. Experts tell you that if you do nothing, the colony will increase its population by 20% every week. So you immediately borrow your friend's pet lizard that eats exactly 400 ants every week.

- (a) (8 points) Give a linear model that describes how the ant population in your kitchen after n weeks P_n evolves from one week to the next.

- (b) (2 points) Is your model from part (a) autonomous? Is it homogeneous?

- (c) (6 points) For what value(s) of P_0 (if any) will the lizard be able to eradicate the ant colony? For what values of P_0 (if any) will the lizard fail to eradicate the ant colony? Justify your answers.

6. Let

$$f(x, y) = x^3 + y^3 - 3x - 12y + 5.$$

- (a) (6 points) Find the partial derivatives f_x and f_y .
 (b) (6 points) Find the critical points of f . That is, find all points (a, b) such that

$$f_x(a, b) = f_y(a, b) = 0.$$

7. Suppose $p = 100$ is a fixed point of the linear model

$$x_{n+1} = ax_n + 80.$$

- (a) (6 points) Find a .
 (b) (2 points) Is the fixed point p stable or unstable?
 (c) (4 points) For what values of x_0 would solutions monotonically decrease?
 (d) (4 points) For what values of x_0 would solutions oscillate above/below the fixed point p ?

8. Consider the following two sets of data.

n	0	1	2	3
x_n	10	70	-110	430

(a) Data set A

n	0	1	2	3
x_n	10	30	150	450

(b) Data set B

One and only one of the data sets above can be modelled precisely by an autonomous linear equation.

- (a) (6 points) Which one? Justify your answer geometrically.
 (b) (8 points) Find the autonomous linear model that fits that data set.