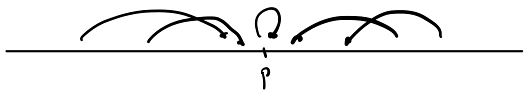
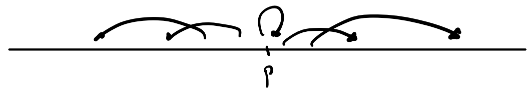
§3.5 Periodic Points & Cycles

So far, fixed point dynamics $x_{n+1} = f(x_n)$

n	x_n
0	p
1	p
2	p
3	p
4	p
5	p
6	p
7	p
8	p
⋮	⋮



STABLE ATTRACTOR



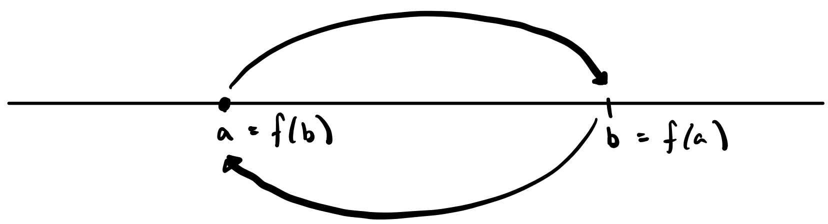
UNSTABLE REPELLER

(ANOTHER EQUILIBRIUM STATE)

$x_{n+1} = f(x_n)$

WE CAN GENERALIZE THE IDEA OF A FIXED POINT:

n	x_n	
0	a	0
1	b	1
2	a	0
3	b	1
4	a	0
5	b	1
6	a	0
7	b	1
8	a	0
⋮	⋮	



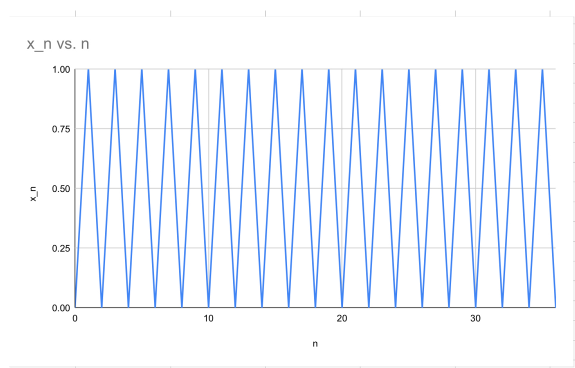
ex. $x_{n+1} = 1 - x_n^a \quad , \; a > 0$

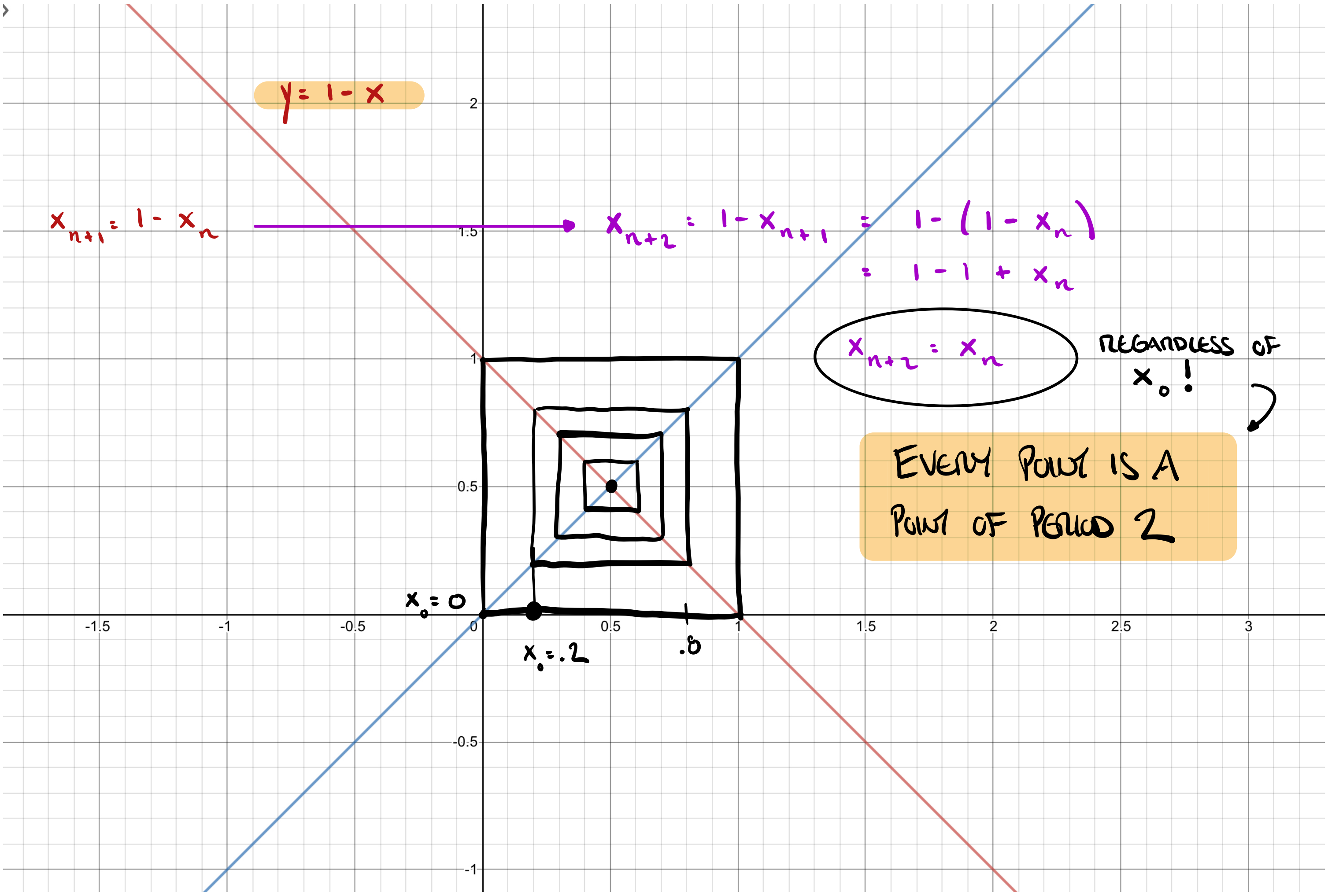
REPEAT!

$$\begin{cases} \text{IF } x_0 = 0 \text{ THEN } x_1 = 1 - x_0^a = 1 - 0 = 1 \\ \text{IF } x_1 = 1 \text{ THEN } x_2 = 1 - x_1^a = 1 - 1 = 0 \end{cases}$$

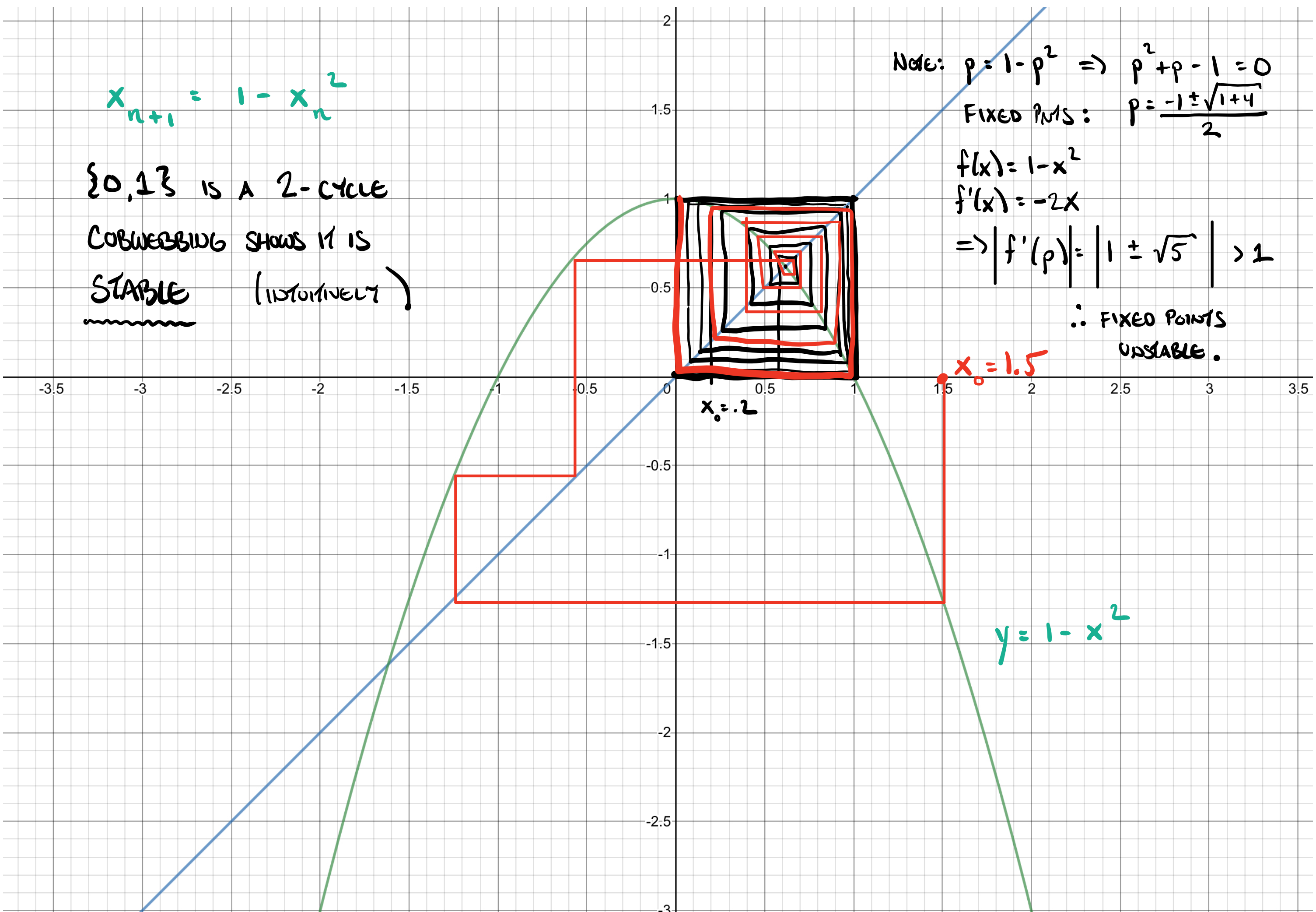
DEFINITION

A pair of distinct points a and b satisfying $f(a) = b$ and $f(b) = a$ is called a **2-cycle** of $x_{n+1} = f(x_n)$, and each point is called a **point of period 2** for $f(x)$.

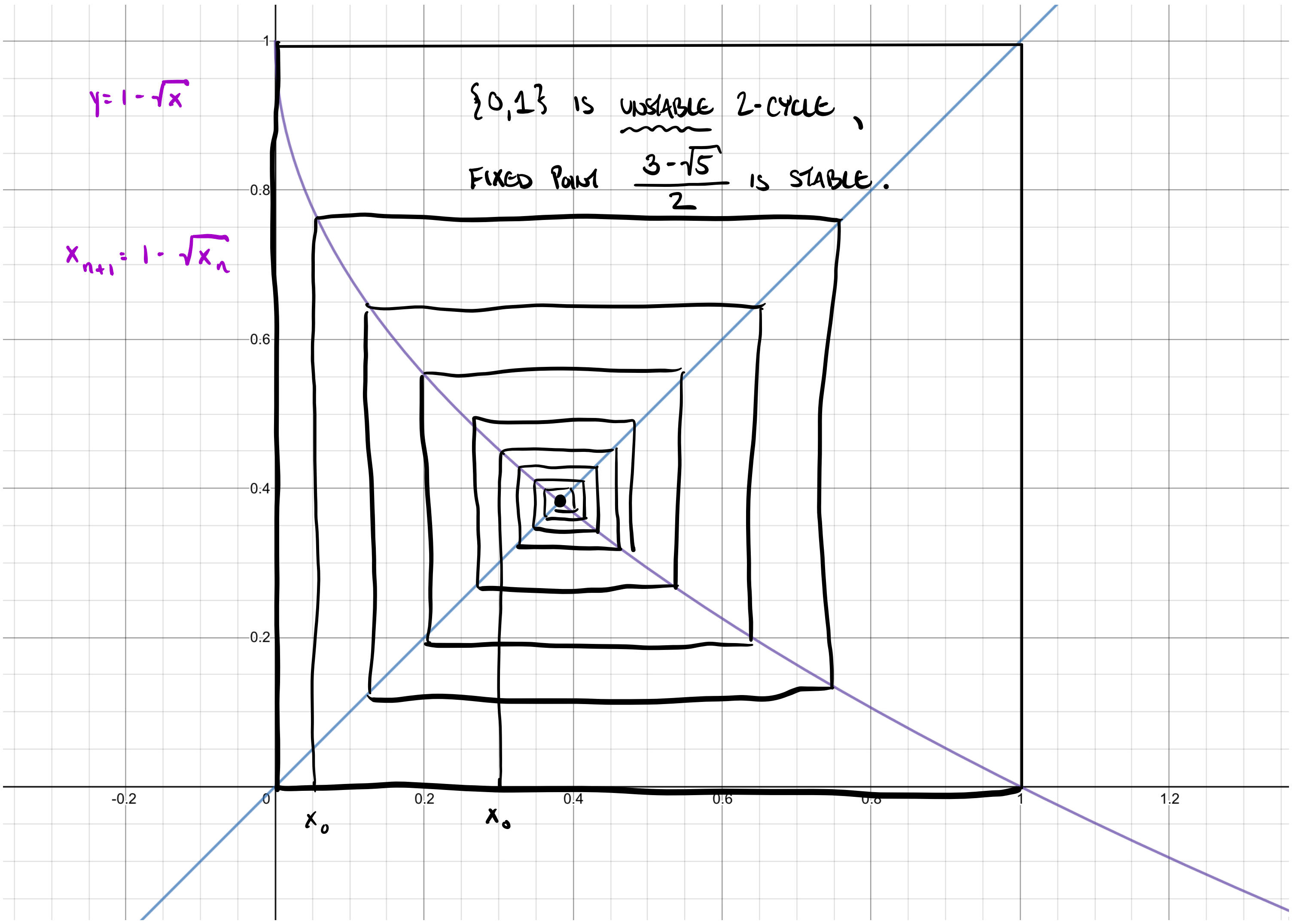




y = 1 - x
$x_{n+1} = 1 - x_n$
$x_{n+2} = 1 - x_{n+1} = 1 - (1 - x_n)$
$= 1 - 1 + x_n$
$x_{n+2} = x_n$
REGARDLESS OF x_0 !
EVERY POINT IS A POINT OF PERIOD 2
$x_0 = 0$
$x_0 = .2$
.8



$x_{n+1} = 1 - x_n^2$
{0,1} IS A 2-CYCLE
COBWEBBING SHOWS IT IS STABLE (INTUITIVELY)
Note: $p = 1 - p^2 \Rightarrow p^2 + p - 1 = 0$
FIXED PNTS: $p = \frac{-1 \pm \sqrt{1+4}}{2}$
$f(x) = 1 - x^2$
$f'(x) = -2x$
$\Rightarrow |f'(p)| = |1 \pm \sqrt{5}| > 1$
$\therefore$ FIXED POINTS UNSTABLE.
$x_0 = 1.5$
$x_0 = .2$
$y = 1 - x^2$



$p = 1 - \sqrt{p}$, $p \le 1$

$\Rightarrow 0 = p^2 - 3p + 1$

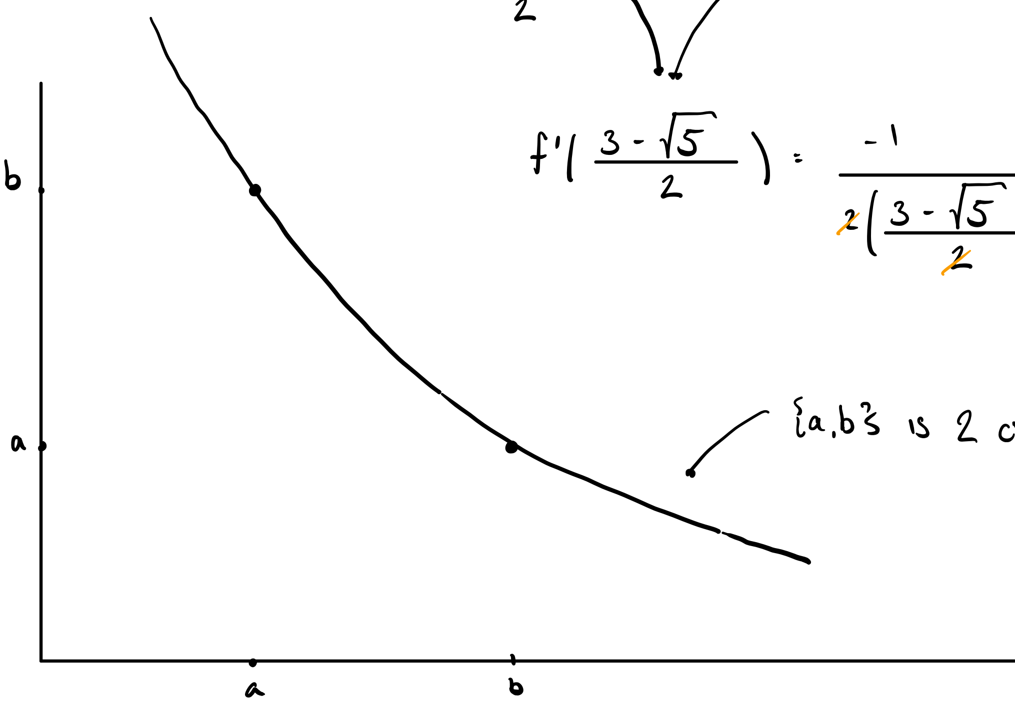
$p = \frac{3-\sqrt{5}}{2}$

$f(x) = 1 - \sqrt{x}$

$f'(x) = -\frac{1}{2\sqrt{x}}$

$$f'\left(\frac{3-\sqrt{5}}{2}\right) = \frac{-1}{\cancel{2}\left(\frac{3-\sqrt{5}}{\cancel{2}}\right)} = \frac{-1}{3-\sqrt{5}}$$

≈ -1.31



$\{a,b\}$ is 2 cycle.

DEFINITION

A pair of distinct points a and b satisfying $f(a) = b$ and $f(b) = a$ is called a **2-cycle** of $x_{n+1} = f(x_n)$, and each point is called a **point of period 2** for $f(x)$.

2 VALUES REPEATED
"LOOP"

IF a & b ARE A 2-CYCLE

THEN BOTH a & b ARE FIXED POINTS OF $f(f(x))$

VERIFY: $f(f(a)) = f(b) = a$ ✓

$f(f(b)) = f(a) = b$ ✓

IF p IS A FIXED POINT OF f: $f(p) = p$, THEN p IS ALSO A FIXED POINT OF $f(f(x))$: $f(f(p)) = f(p) = p$. BUT THE PAIR (p, p) IS NOT A 2-CYCLE!

A POINT OF A 2-CYCLE OF $x_{n+1} = f(x_n)$ IS A FIXED POINT OF $f(f(x))$ THAT IS NOT A FIXED POINT OF f.

FIXED POINTS OF $f(x)$:

$x^2 - 2x = x$

$x^2 - 3x = 0$

$x(x-3) = 0$

$x = 0, 3$

EX. FIND ALL 2-CYCLES (IF ANY) OF $x_{n+1} = x_n^2 - 2x_n$

2-CYCLE: $x_{n+2} = f(x_{n+1}) = f(f(x_n))$

IF THESE ARE THE SAME THEN x_n IS POTENTIALLY ONE # IN A 2-CYCLE

FIXED POINT OF $f(f(x))$: $f(f(x)) = x$ ← KNOWN SOL'NS ARE FIXED PNTS OF f: $x = 0, 3$

$f(x^2 - 2x) = x$

$(x^2-2x)^2 - 2(x^2-2x) = x$

$x^4 - 4x^3 + 4x^2 - 2x^2 + 4x = x$

$x^4 - 4x^3 + 2x^2 + 3x = 0$ ← KNOWN ROOTS: $x = 0, 3$ => FACTORS OF THIS POLYNOMIAL $x(x-3)$

$x(x^3 - 4x^2 + 2x + 3) = 0$

$$\begin{array}{r}
x^2 - x - 1 \\
x - 3 \,\overline{\big)\, x^3 - 4x^2 + 2x + 3} \\
-\left(x^3 - 3x^2\right) \\
\hline
-x^2 + 2x + 3 \\
-\left(-x^2 + 3x\right) \\
\hline
-x + 3 \\
-\left(-x + 3\right) \\
\hline
0
\end{array}$$

$f(f(x)) = x$

3 sol'ns

$(x-1)^2$

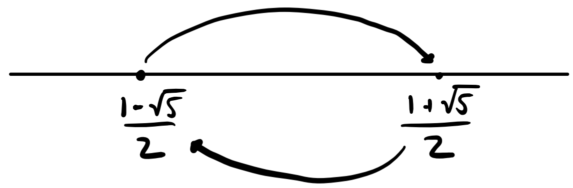
$x(x-3)(x^2 - x - 1) = 0$

$x = 0$ FIXED

$x = 3$ FIXED

$x = \dfrac{1 \pm \sqrt{1+4}}{2} = \dfrac{1 \pm \sqrt{5}}{2}$

2-cycle!



RULE

To find a 2-cycle of a polynomial $f(x)$, first divide $\underbrace{f(f(x)) - x}_{\text{Fixed pnt. Eq for } f(f(x))}$ by $x - p$ for each fixed point p. This will reduce the degree of the polynomial involved and simplify solving $f(f(x)) - x = 0$. The same may work when $f(x)$ is not a polynomial but if $f(f(x)) - x = 0$ can be converted into a polynomial equation.

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e.g.: $f(x) = \frac{2}{x} - x$

$0 = f(f(x)) - x = \dfrac{2}{\left(\frac{2}{x} - x\right)} - \left(\frac{2}{x} - x\right) - x = 0$

clear denom. mult by LCD $\Rightarrow$ poly eq.

Stability of 2-cycles

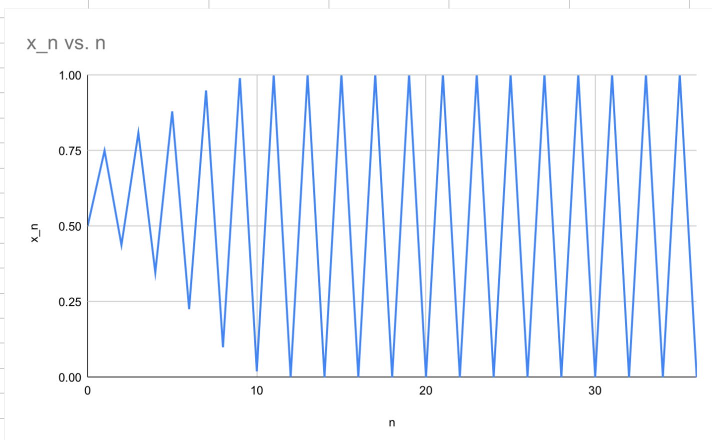
n	x_n
0	
1	
2	
3	
4	
5	
6	
7	
8	

$x_{2n} \to a$

$x_{2n+1} \to b$

STABLE 2-CYCLE

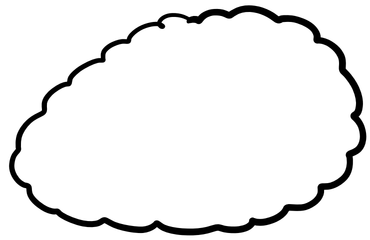
n	x_n	x_{n+1} = 1 - (x_n)^p	p
			2
0	0.5		
1	0.75		
2	0.4375		
3	0.80859375		
4	0.3461761475		
5	0.8801620749		
6	0.2253147219		
7	0.9492332761		
8	0.09895618752		
9	0.990207673		
10	0.01948876443		
11	0.9996201881		
12	0.000759479620		
13	0.9999994232		
14	0.000001153618		
15	1		
16	0		
17	1		
18	0		
19	1		
20	0		



RECALL: A POINT OF A 2-CYCLE OF $x_{n+1} = f(x_n)$

IS A FIXED POINT OF $f(f(x))$ THAT IS NOT

A FIXED POINT OF f.



DEFINITION

A 2-cycle of $x_{n+1} = f(x_n)$ is **locally stable** if each point of the cycle is a stable fixed point of $f(f(x))$. Otherwise, the 2-cycle is **unstable.** Either both points are stable or both are unstable.

NOTE: IF EVEN ITERATES X_{2n} GET CLOSER & CLOSER TO a

THEN THE ODD ITERATES X_{2n+1}

MUST GET CLOSER & CLOSER TO $b = f(a)$

i.e. f IS CONTINUOUS AT a.

SUPPOSE $\{a, b\}$ IS A 2-CYCLE $(f(a) = b,\ f(b) = a)$.

IT IS STABLE IF $\left| \frac{d}{dx}\left[f(f(x)) \right]_{x=\underset{b}{a}} \right| < 1$

$$\left| f'(f(\underset{b}{a}))\, f'(\underset{b}{a}) \right| < 1$$

$$\left| f'(b)\, f'(a) \right| < 1 \quad \text{STABLE 2-CYCLE}$$

SIMILARLY, $\left| f'(b)\, f'(a) \right| > 1$ UNSTABLE 2 CYCLE

THEOREM *Stability of 2-Cycles*

If a and b are the two points of a 2-cycle of $x_{n+1} = f(x_n)$, then the cycle is locally stable if $|f'(a)f'(b)| < 1$ or unstable if $|f'(a)f'(b)| > 1$. ■

ex. DETERMINE THE STABILITY OF 2-CYCLE $\{0, 1\}$ OF $x_{n+1} = 1 - x_n^2$.

$f(x) = 1 - x^2$, $f'(x) = -2x$

$|f'(0)\, f'(1)| = |(0)(-2)| = 0 < 1$

$\therefore$ $\{0, 1\}$ IS A STABLE 2-CYCLE.

ex. DETERMINE THE STABILITY OF 2-CYCLE OF $x_{n+1} = x_n^2 - 2x_n$.

2-CYCLE $\left\{ \frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2} \right\}$

$f(x) = x^2 - 2x$ $\qquad$ $f'(x) = 2x - 2$

$$\left| f'\left(\frac{1-\sqrt{5}}{2}\right) f'\left(\frac{1+\sqrt{5}}{2}\right) \right| = \left| \left[1 - \sqrt{5} - 2\right]\left[1 + \sqrt{5} - 2\right] \right|$$

$$\left| \left[-1 - \sqrt{5}\right]\left[-1 + \sqrt{5}\right] \right| = \left| (-1)^2 - 5 \right| = |-4| = 4 > 1$$

$\therefore$ 2-CYCLE $\left\{ \frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2} \right\}$ IS UNSTABLE.

GENERALIZATION TO m-CYCLES

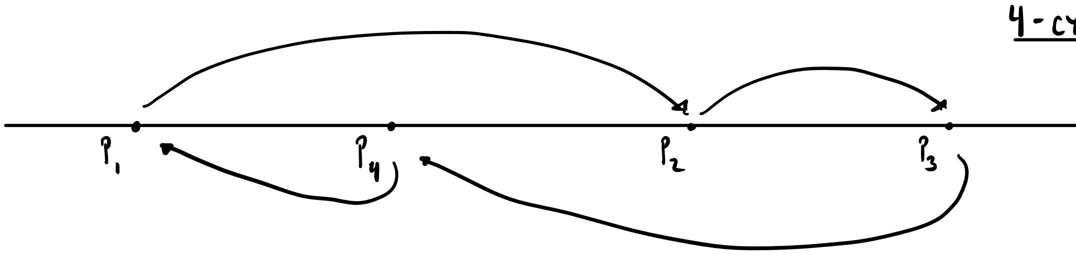
DEFINITION LOOP (LONGER)

A set of m distinct points $p_1, p_2, \ldots, p_m$ satisfying

$$f(p_1) = p_2, \quad f(p_2) = p_3, \quad \ldots, \quad f(p_{m-1}) = p_m, \quad f(p_m) = p_1$$

is called an ***m*-cycle** of $x_{n+1} = f(x_n)$. Each point p_i of an m-cycle is called a **point of period *m*** for $f(x)$.

4-CYCLE



POINT OF PERIOD m IS A FIXED POINT OF $f^m(x)$
THAT IS NOT FIXED BY $f(x), f^2(x), \ldots, f^{m-1}(x)$.

DEFINITION

An m-cycle of $x_{n+1} = f(x_n)$ is **locally stable** if each point of the cycle is a locally stable fixed point of $f^m(x)$. Otherwise it is **unstable.** Either all points of an m-cycle are stable or all are unstable.

EITHER ALL POINTS OF m-CYCLE ARE ATTRACTORS, OR
ALL POINTS OF m-CYCLE ARE REPELLERS.

WHY? f IS CONTINUOUS: $\lim_{x \to p_1} f(x) = f(p_1) = p_2 \Rightarrow \lim_{x \to p_1} f \circ f(x) = f \circ f(p_1) = p_3$

$\Rightarrow$ ETC.

CLASSIFY STABILITY OF m-CYCLE:

POINT OF PERIOD m IS A FIXED POINT OF $f^m(x)$

$\Rightarrow$ IF p IS A POINT OF PERIOD m, THEN p IS STABLE IF

$$\left| \frac{d}{dx}\left[f^m(x)\right]\Big|_{x=p} \right| < 1$$

DERIV. AT p

$$f(f(f\cdots(x))) = f'(f^{m-1}(p))\, f'(f^{m-2}(p)) \cdots f'(f(p))\, f'(p)$$

m POINTS OF CYCLE

$$\left| f'(p_1)\, f'(p_2) \cdots f'(p_m) \right| < 1 \checkmark$$

THEOREM *Stability of M-Cycles*

An m-cycle $p_1, p_2, \ldots, p_m$ of $x_{n+1} = f(x_n)$ is locally stable if

$$|f'(p_1)\cdot f'(p_2)\cdot f'(p_3)\cdot \cdots \cdot f'(p_m)| < 1$$

or unstable if

$$|f'(p_1)\cdot f'(p_2)\cdot f'(p_3)\cdot \cdots \cdot f'(p_m)| > 1 \qquad \blacksquare$$

ex. GIVEN THAT $\frac{7}{9}$ IS A POINT OF PERIOD 3 FOR $x_{n+1} = 1 - 2|x_n|$, FIND THE OTHER POINTS IN ITS 3-CYCLE & DETERMINE THE STABILITY OF THE 3-CYCLE.

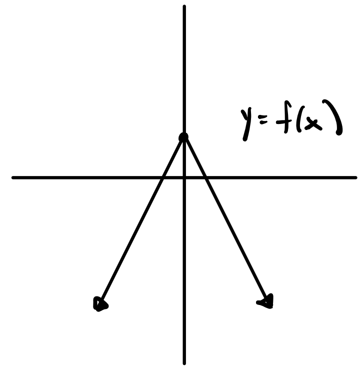
$$f\left(\tfrac{7}{9}\right) = 1 - 2\left|\tfrac{7}{9}\right| = \frac{9}{9} - \frac{14}{9} = -\frac{5}{9}$$

$$f\left(-\tfrac{5}{9}\right) = 1 - 2\left|-\tfrac{5}{9}\right| = \frac{9}{9} - \frac{10}{9} = -\frac{1}{9}$$

$$f\left(-\tfrac{1}{9}\right) = 1 - 2\left|-\tfrac{1}{9}\right| = \frac{9}{9} - \frac{2}{9} = \frac{7}{9}$$

$$\begin{tikzcd} \frac{7}{9} \arrow[r, "f"] & -\frac{5}{9} \arrow[r, "f"] & -\frac{1}{9} \arrow[ll, bend left, "f"] \\ p_1 & p_2 & p_3 \end{tikzcd}$$

3-CYCLE IS STABLE IFF $|f'(p_1)f'(p_2)f'(p_3)| < 1$.



$$f(x) = 1 - 2|x| = \begin{cases} 1 - 2x & \text{IF } x \geq 0 \\ 1 - 2(-x) = 1 + 2x & \text{IF } x < 0 \end{cases}$$

$$f'(x) = \begin{cases} -2 & \text{IF } x > 0 \\ \text{UND.} & \text{IF } x = 0 \\ 2 & \text{IF } x < 0 \end{cases}$$

$$|f'(p_1)f'(p_2)f'(p_3)|$$

$$\left|f'\left(\tfrac{7}{9}\right) f'\left(-\tfrac{5}{9}\right) f'\left(-\tfrac{1}{9}\right)\right|$$

$$|(-2)(2)(2)| = 8 > 1$$

$$\Rightarrow \boxed{\text{UNSTABLE}}$$