1. Evaluate each of the following integrals.

 (a) (6 points) $\int e^{2x} \sinh x \, dx$

INTEGRATION BY PARTS:

$u = e^{2x}$ $v = \cosh x$

$du = 2e^{2x}\,dx$ $dv = \sinh x \, dx$

AGAIN: $u = e^{2x}$ $v = \sinh x$

$du = 2e^{2x}\,dx$ $dv = \cosh x \, dx$

$$\int e^{2x} \sinh x \, dx = e^{2x}\cosh x - 2\int e^{2x}\cosh x \, dx$$

$$\int e^{2x} \sinh x \, dx = e^{2x}\cosh x - 2\left(e^{2x}\sinh x - 2\int e^{2x}\sinh x \, dx\right)$$

$$-3\int e^{2x} \sinh x \, dx = e^{2x}\cosh x - 2e^{2x}\sinh x$$

$$\int e^{2x} \sinh x \, dx = \boxed{\frac{1}{3}e^{2x}\left(2\sinh x - \cosh x\right) + C}$$

ALT:

$$\int e^{2x}\sinh x \, dx = \frac{1}{2}e^{2x}\sinh x - \frac{1}{2}\left(\frac{1}{2}e^{2x}\cosh x - \frac{1}{2}\int e^{2x}\sinh x \, dx\right)$$

SAME

 (b) (6 points) $\int \frac{\tan^5(\ln x)\sec^7(\ln x)}{x}\,dx$

u-SUB: $u = \ln x$ $du = \frac{1}{x}\,dx$

$$\rightsquigarrow \int \tan^4 u \sec^6 u \sec u \tan u \, du$$

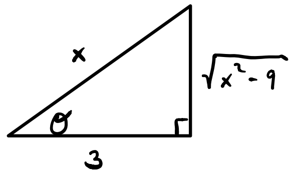
W-SUB: $w = \sec u$, $dw = \sec u \tan u \, du$

$$\rightsquigarrow \int (w^2-1)^2 w^6 \, dw = \int w^{10} - 2w^8 + w^6 \, dw = \frac{1}{11}w^{11} - \frac{2}{9}w^9 + \frac{1}{7}w^7 + C$$

$$\rightsquigarrow \frac{1}{11}\sec^{11}u - \frac{2}{9}\sec^9 u + \frac{1}{7}\sec^7 u + C$$

$$\rightsquigarrow \boxed{\frac{1}{11}\sec^{11}(\ln x) - \frac{2}{9}\sec^9(\ln x) + \frac{1}{7}\sec^7(\ln x) + C}$$

(c) (6 points) $\displaystyle\int_{3}^{3\sqrt{2}} \frac{\sqrt{x^2-9}}{x^3}\,dx$



$\sec\theta = \frac{x}{3}$ $\quad \cos\theta = \frac{3}{x}$

TRIG - SUB: $\quad x = 3\sec\theta$

$dx = 3\sec\theta\tan\theta\,d\theta$

$x = 3\sqrt{2} \Rightarrow \cos\theta = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \Rightarrow \theta = \pi/4$

$x = 3 \Rightarrow \cos\theta = 1 \Rightarrow \theta = 0$

$$\rightsquigarrow \int_0^{\pi/4} \frac{\sqrt{9\sec^2\theta - 9}}{27\sec^3\theta}\, 3\sec\theta\tan\theta\,d\theta$$

$$= \int_0^{\pi/4} \frac{3\tan\theta}{27\sec^3\theta}\, 3\sec\theta\tan\theta\,d\theta = \frac{1}{3}\int_0^{\pi/4} \frac{\tan^2\theta}{\sec^2\theta}\,d\theta = \frac{1}{3}\int_0^{\pi/4} \sin^2\theta\,d\theta$$

$$= \frac{1}{6}\int_0^{\pi/4} 1 - \cos 2\theta\,d\theta = \frac{1}{6}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi/4} = \frac{1}{6}\left[\left(\frac{\pi}{4} - 0\right) - \frac{1}{2}\left(1 - 0\right)\right]$$

$$= \boxed{\frac{1}{6}\left(\frac{\pi}{4} - \frac{1}{2}\right) = \frac{\pi - 2}{24}}$$

(d) (6 points) $\displaystyle\int \frac{10}{(x-1)(x^2+9)}\,dx$

PARTIAL FRACTIONS: $\quad \dfrac{10}{(x-1)(x^2+9)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+9}$

$$\Rightarrow 10 = A(x^2+9) + (Bx+C)(x-1) = Ax^2 + 9A + Bx^2 - Bx + Cx - C$$

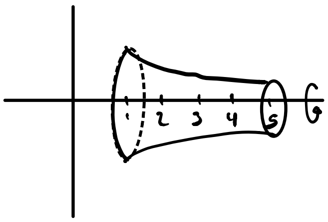
$$0x^2 + 0x + 10 = (A+B)x^2 + (-B+C)x + 9A - C$$

$$\Rightarrow \begin{aligned} A + B &= 0 \\ -B + C &= 0 \\ 9A - C &= 10 \end{aligned} \qquad A = 1,\; B = -1,\; C = -1$$

$$\therefore \int \frac{10}{(x-1)(x^2+9)}\,dx = \int \frac{1}{x-1} - \frac{x}{x^2+9} - \frac{1}{x^2+9}\,dx$$

$$= \boxed{\ln|x-1| - \frac{1}{2}\ln|x^2+9| - \frac{1}{3}\tan^{-1}\frac{x}{3} + C}$$

2. (6 points) Use Simpson's Rule with $n = 4$ to approximate the area of the surface obtained by rotating the curve $y = 1/x$, $1 \le x \le 5$ around the x-axis. Leave your answer as a product/sum of terms.



$$S = \int_1^5 2\pi y \, ds \quad , \quad y = \frac{1}{x} \quad , \quad ds = \sqrt{\left(\frac{dy}{dx}\right)^2 + 1} \, dx$$

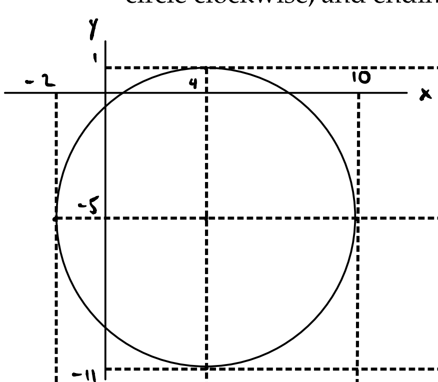
$$= \int_1^5 2\pi \frac{1}{x} \sqrt{\frac{1}{x^4} + 1} \, dx$$

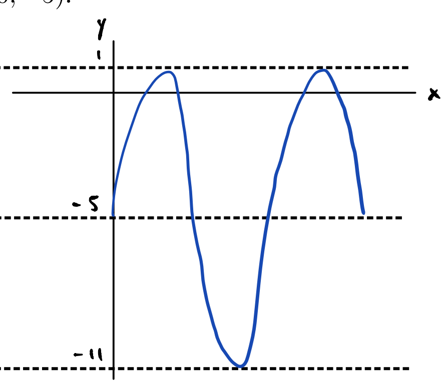
Simpson's Rule: $n = 4$

$$S \approx \frac{\Delta x}{3} \left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4) \right]$$

$$S \approx \boxed{\frac{1}{3} \cdot 2\pi \left[\sqrt{1+1} + 4 \cdot \frac{1}{2} \sqrt{\frac{1}{2^4} + 1} + 2 \cdot \frac{1}{3} \sqrt{\frac{1}{3^4} + 1} + 4 \cdot \frac{1}{4} \sqrt{\frac{1}{4^4} + 1} + \frac{1}{5} \sqrt{\frac{1}{5^4} + 1} \right]}$$

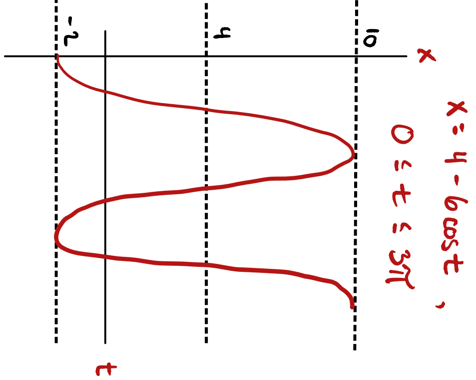
3. (5 points) Give parametric equations to describe the motion of a particle that moves in the xy-plane along the circle with center $(4, -5)$ and radius 6 beginning at $(-2, -5)$, traveling 1.5 times around the circle clockwise, and ending at $(10, -5)$.





$y = -5 + 6 \sin t$

$0 \le t \le 3\pi$



$$\boxed{\begin{array}{l} x = 4 - 6\cos t \\ y = -5 + 6\sin t \\ 0 \le t \le 3\pi \end{array}}$$

4. Let C be the curve given paramtrically by

$$x = t^3, \quad y = \frac{3t^2}{2}, \quad 0 \le t \le \sqrt{3}$$

(a) (5 points) Find the arclength of the curve C.

$$L = \int_0^{\sqrt{3}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{\sqrt{3}} \sqrt{(3t^2)^2 + (3t)^2}\, dt$$

$$= \int_0^{\sqrt{3}} 3t\sqrt{t^2+1}\, dt \qquad u = t^2+1, \quad \frac{3}{2}du = t\,dt$$

$$\rightsquigarrow \frac{3}{2}\int_1^4 u^{1/2}\, du = u^{3/2}\Big|_1^4 = \boxed{4^{3/2} - 1 = 7}$$

(b) (5 points) Give an equation for the line tangent to C at $(1/8, 3/8)$.

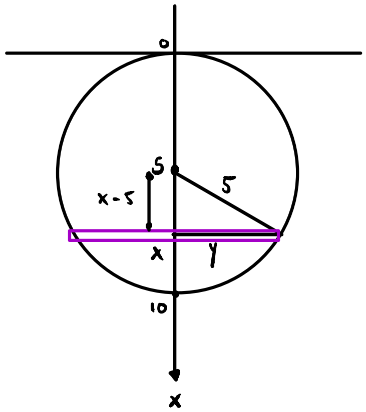
$$x = t^3 = 1/8 \Rightarrow t = \frac{1}{2}$$

Slope: $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{3t}{3t^2} = \dfrac{1}{t}$

$$\Rightarrow \frac{dy}{dx}\Big|_{t=\frac{1}{2}} = 2$$

$$\therefore \boxed{y - \frac{3}{8} = 2\left(x - \frac{1}{8}\right) \text{ or } y = 2x + \frac{1}{8}}$$

5. (7 points) An oil truck carries oil with a density of 55 lbs/ft^3 in a horizontal cylindrical tank with a 10 ft diameter. Setup *but do not evaluate* a definite integral equal to the hydrostatic force exerted by the oil on one end of the tank when the tank is full.



FORCE = PRESSURE × SURFACE AREA

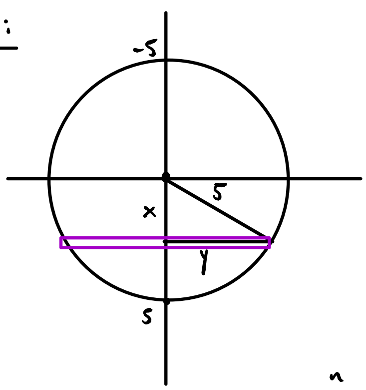
FORCE = DENSITY × DEPTH × $\underbrace{\text{SURFACE AREA}}$

$$2y\Delta x = 2\sqrt{5^2-(x-5)^2}\,\Delta x$$

$$F = \lim_{n\to\infty}\sum_{i=1}^{n}(55\ \text{lbs}/\text{ft}^3)(x\ \text{ft})\left(2\sqrt{5^2-(x-5)^2}\,\Delta x\ \text{ft}^2\right)$$

$$F = \int_0^{10} 110x\sqrt{25-(x-5)^2}\,dx = \boxed{\int_0^{10} 110x\sqrt{10x-x^2}\,dx}$$

Alt:

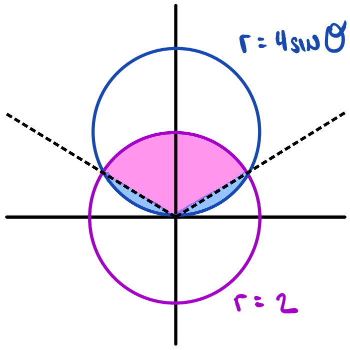


$$F = \lim_{n\to\infty}\sum_{i=1}^{n}(55\ \text{lbs}/\text{ft}^3)(x+5)\left(2\sqrt{5^2-x^2}\,\Delta x\ \text{ft}^2\right)$$

$$\boxed{F = \int_{-5}^{5} 110(x+5)\sqrt{25-x^2}\,dx}$$

6. (7 points) Sketch and find the area of the region that lies inside both of the following polar curves.

$$r = 4\sin\theta, \quad r = 2$$



INTERSECTIONS: $4\sin\theta = 2$

$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \ldots$

BY SYMMETRY, AREA $= 2\int_0^{\pi/6} \frac{1}{2}(4\sin\theta)^2\,d\theta + 2\int_{\pi/6}^{\pi/2} \frac{1}{2}(2)^2\,d\theta$

$$= 8\int_0^{\pi/6} 1 - \cos 2\theta\,d\theta + 4\int_{\pi/6}^{\pi/2} d\theta$$

$$= 8\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi/6} + 4\theta\Big|_{\pi/6}^{\pi/2}$$

$$= 8\left[\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right] + \frac{4\pi}{3} = \boxed{\frac{8\pi}{3} - 2\sqrt{3}}$$

7. Determine whether each of the following is convergent or divergent. If it is convergent, find its sum. If it is diverent, explain why.

(a) (5 points) $\sum_{n=1}^{\infty}(-1)^n \frac{3^{n+2}}{2^{2n}}$

$$= \sum_{n=1}^{\infty} (-1)^n 3^2 \left(\frac{3}{2^2}\right)^n = \sum_{n=1}^{\infty} 9\left(-\frac{3}{4}\right)^n$$

Geometric Series.

$$= \sum_{n=0} 9\left(-\frac{3}{4}\right)\left(-\frac{3}{4}\right)^n = \sum_{n=0}^{\infty}\left(-\frac{27}{4}\right)\left(-\frac{3}{4}\right)^n$$

$$= \frac{-\frac{27}{4}}{1-\left(-\frac{3}{4}\right)} = -\frac{27}{4}\cdot\frac{4}{7} = \boxed{-\frac{27}{7}}$$

(b) (5 points) $\sum_{n=1}^{\infty} \ln \frac{n}{n+1}$

Let $S_K = \sum_{n=1}^{K} \ln \frac{n}{n+1} = \sum_{n=1}^{K} \left(\ln(n) - \ln(n+1)\right)$

$$S_K = (\ln(1) - \cancel{\ln(2)}) + (\cancel{\ln(2)} - \cancel{\ln(3)}) + \ldots + (\cancel{\ln(K)} - \ln(K+1))$$

0

$\therefore S_K = -\ln(K+1)$

$$\sum_{n=1}^{\infty} \ln\frac{n}{n+1} = \lim_{K\to\infty} S_K = \lim_{K\to\infty} -\ln(K+1) = -\infty$$

$\boxed{\text{DIVERGES}}$

8. Determine whether each of the following is convergent or divergent and explain why.

(a) (5 points) $\displaystyle\int_1^\infty \frac{1+x}{x^2-e^{-x}}\,dx$

Note: $\dfrac{1+x}{x^2-e^{-x}} > \dfrac{x}{x^2} = \dfrac{1}{x}$

$\Rightarrow \displaystyle\int_1^\infty \frac{1+x}{x^2-e^{-x}}\,dx > \int_1^\infty \frac{1}{x}\,dx = \infty$ DIVERGES BY p-TEST WITH $p = 1 \leq 1$.

$\therefore$ $\boxed{\text{DIVERGES BY DIRECT COMPARISON THEOREM}}$

(b) (5 points) $\displaystyle\sum_{n=1}^\infty (-1)^n \left(\frac{1}{n^2}\right)^{1/\sqrt{n}}$

ALTERNATING SERIES $\displaystyle\sum_{n=1}^\infty (-1)^n u_n$ CONVERGE IF & ONLY IF $\displaystyle\lim_{n\to\infty} u_n = 0$.

$\displaystyle\lim_{n\to\infty} \left(\frac{1}{n^2}\right)^{1/\sqrt{n}} : 0^0$ INDETERMINATE FORM

$= \text{EXP}\left[\displaystyle\lim_{n\to\infty} \frac{\ln\left(\frac{1}{n^2}\right)}{\sqrt{n}}\right] = \text{EXP}\left[\displaystyle\lim_{n\to\infty} \frac{-2\ln n}{\sqrt{n}}\right] =$

$\xrightarrow{\text{L'HO. RULE}} \text{EXP}\left[\displaystyle\lim_{n\to\infty} \frac{-2}{n}\cdot 2\sqrt{n}\right] = \displaystyle\lim_{n\to\infty} e^{-4/\sqrt{n}} = 1$

$\therefore$ $\boxed{\text{SERIES DIVERGES BY DIVERGENCE TEST (i.e. ALT. SERIES TEST)}}$

9. Consider the power series $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{\sqrt{n+3}}(x-3)^n$.

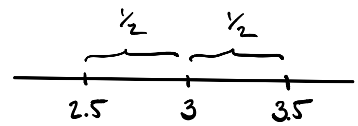
(a) (5 points) Find the radius of convergence.

RATIO TEST: $\displaystyle \rho(x) = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{2^{n+1}(x-3)^{n-1}}{\sqrt{n+1+3}} \cdot \frac{\sqrt{n+3}}{2^n (x-3)^n} \right|$

$\displaystyle = \lim_{n\to\infty} \sqrt{\frac{n+3}{n+4}} \cdot 2|x-3| = 2|x-3|$

SERIES CONVERGES ABSOLUTELY WHEN $\rho(x) < 1$: $2|x-3| < 1 \Rightarrow |x-3| < \frac{1}{2}$

(& DIVERGES WHEN $\rho(x) > 1$: $|x-3| > \frac{1}{2}$)



$\therefore$ RADIUS OF CONVERGENCE $\boxed{R = \frac{1}{2}}$

(b) (4 points) Find the interval of convergence.

WE JUST NEED TO TEST WHAT HAPPENS WHEN $\rho(x) = 1$: $|x-3| = \frac{1}{2}$

i.e. WHEN $x = \frac{5}{2}$ & $x = \frac{7}{2}$.

$x = \frac{5}{2}$: $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{\sqrt{n+3}}(x-3)^n = \sum_{n=0}^{\infty} \frac{2^n(-\frac{1}{2})^n}{\sqrt{n+3}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+3}}$

CONVERGES BY ALT. SERIES TEST.

$x = \frac{7}{2}$: $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{\sqrt{n+3}}(x-3)^n = \sum_{n=0}^{\infty} \frac{2^n(\frac{1}{2})^n}{\sqrt{n+3}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+3}}$

$\displaystyle = \sum_{n=3}^{\infty} \frac{1}{\sqrt{n}}$ DIVERGES BY p-TEST WITH $p = \frac{1}{2} \leq 1$

$\therefore$ INTERVAL OF CONVERGENCE : $\boxed{[\,^5/_2 \,,\, ^7/_2\,)}$