Exam 2

1. (12 points) Find the arclength of the curve

$$y = \frac{1}{4}x^2 - \frac{1}{2}\ln x, \quad 1 \le x \le 2.$$

$$L = \int_1^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_1^2 \sqrt{1 + \left(\frac{x}{2} - \frac{1}{2x}\right)^2}\, dx$$

Note: $1 + \left(\frac{x}{2} - \frac{1}{2x}\right)^2 = 1 + \frac{x^2}{4} - \frac{1}{2} + \frac{1}{4x^2} = \frac{x^2}{4} + \frac{1}{2} + \frac{1}{4x^2} = \left(\frac{x}{2} + \frac{1}{2x}\right)^2$

$$L = \int_1^2 \sqrt{\left(\frac{x}{2} + \frac{1}{2x}\right)^2}\, dx = \int_1^2 \frac{x}{2} + \frac{1}{2x}\, dx = \frac{1}{4}x^2 + \frac{1}{2}\ln x \Big|_1^2$$

$$= \left(1 + \frac{1}{2}\ln 2\right) - \left(\frac{1}{4} + 0\right) = \boxed{\frac{3}{4} + \frac{1}{2}\ln 2}$$

2. Consider the curve

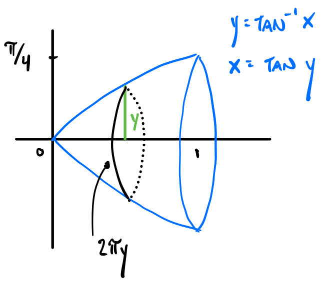
$$y = \tan^{-1} x, \quad 0 \le x \le 1.$$

Setup an integral for the area of each surface of revolution described below. *Do not evaluate the integrals.*

(a) (10 points) The curve is rotated about the x-axis.

(b) (10 points) The curve is rotated about the y-axis.

(a)

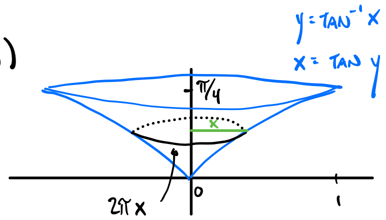


$$A = \int 2\pi y\, ds$$

$$= \int 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \quad \text{or} \quad \int 2\pi y \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$A = \boxed{\int_0^1 2\pi \tan^{-1} x \sqrt{1 + \left(\frac{1}{1+x^2}\right)^2}\, dx} \quad \text{or} \quad \boxed{\int_0^{\pi/4} 2\pi y \sqrt{1 + \left(\sec^2 y\right)^2}\, dy}$$

(b)

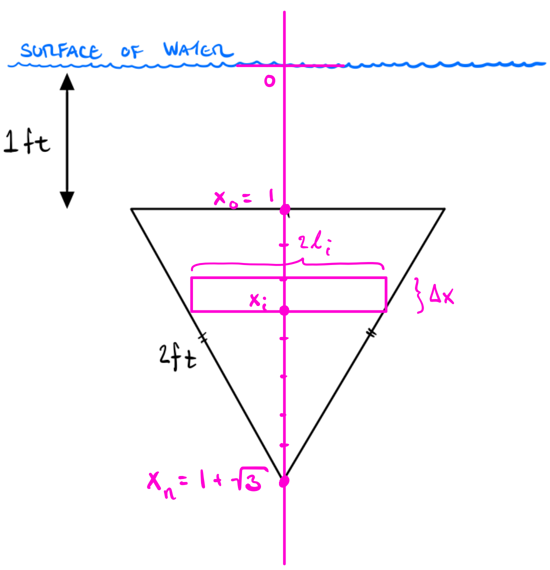


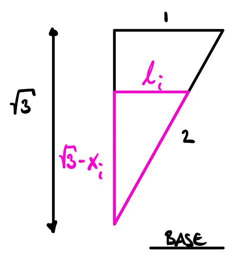
$$A = \int 2\pi x \, ds$$

$$= \int 2\pi x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \quad \text{or} \quad \int 2\pi x \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

$$A = \boxed{\int_0^1 2\pi x \sqrt{1 + \left(\frac{1}{1+x^2}\right)^2}\, dx} \quad \text{or} \quad \boxed{\int_0^{\pi/4} 2\pi \tan y \sqrt{1 + \left(\sec^2 y\right)^2}\, dy}$$

3. (16 points) A vertical plate in the shape of an equilateral triangle with side length 2 ft is submerged in water 1 ft below the surface as in figure 1a. Approximate the hydrostatic force against one side of the plate by a Riemann sum. Then express the force as an integral and evaluate it. Let δ equal the weight density of water per cubic ft, and leave your answer in terms of δ.





SIMILAR TRIANGLES

$$\frac{\text{BASE}}{\text{HEIGHT}} = \frac{\ell_i}{1+\sqrt{3}-x_i} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \ell_i = \frac{1}{\sqrt{3}}\left(1+\sqrt{3}-x_i\right)$$

FORCE ON i^{TH} HORIZONTAL STRIP $F_i \approx (\text{AREA})(\text{DEPTH})(\text{WEIGHT DENSITY})$

$$\approx \left(2 \cdot \frac{1}{\sqrt{3}}\left(1+\sqrt{3}-x_i\right)\Delta x\right)(x_i)(\delta)$$

TOTAL FORCE $\boxed{F \approx \sum_{i=1}^{n} \frac{2}{\sqrt{3}} \delta x_i \left(1+\sqrt{3}-x_i\right) \Delta x}$

$$F = \int_1^{1+\sqrt{3}} \frac{2}{\sqrt{3}} \delta x \left(1+\sqrt{3}-x\right) dx = \frac{2\delta}{\sqrt{3}} \int_1^{1+\sqrt{3}} (1+\sqrt{3})x - x^2 \, dx = \frac{2\delta}{\sqrt{3}} \left[\frac{1+\sqrt{3}}{2} x^2 - \frac{1}{3} x^3\right]_1^{1+\sqrt{3}}$$

$$= \boxed{\frac{2\delta}{\sqrt{3}} \left[\frac{1+\sqrt{3}}{2}(1+\sqrt{3})^2 - \frac{1}{3}(1+\sqrt{3})^3 - \frac{1+\sqrt{3}}{2} + \frac{1}{3}\right]}$$

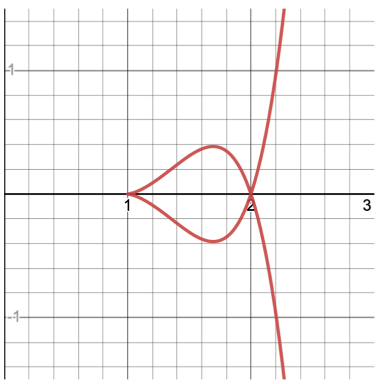
4. The x- and y-coordinates of a moving particle at time t are given by the parametric equations

$$\begin{cases} x = t^{2/3} + 1 \\ y = t^3 - t \end{cases},$$

and the path of the particle is shown in figure 1b. Observe that the particle passes through the point $(2, 0)$ twice.

(a) (6 points) Find the times t_1 and t_2 at which the particle is at the point $(2, 0)$.

(b) (10 points) Give equations for both tangent lines to the curve at the point $(2, 0)$.



(a) $x = t^{2/3} + 1 = 2$

$y = t^3 - t = 0$ $\Rightarrow$ $\boxed{t = \pm 1}$

(b) $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{3t^2 - 1}{\frac{2}{3} t^{-1/3}}$

$\therefore \left.\dfrac{dy}{dx}\right|_{t=1} = \dfrac{3-1}{2/3} = 3$

$\left.\dfrac{dy}{dx}\right|_{t=-1} = \dfrac{3-1}{-2/3} = -3$

TANGENT LINE WHEN $t = 1$: $y = 3(x-2)$

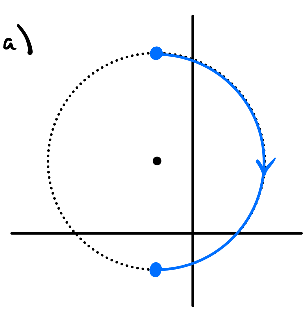
TANGENT LINE WHEN $t = -1$: $y = -3(x-2)$

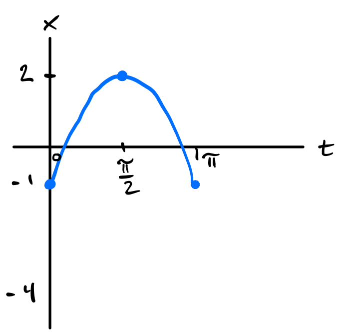
5. Give parametric equations that describe the motion of a particle that moves around the circle with center $(-1, 2)$ and radius 3 in the manner described. Remember to specify the domain of the parametric eequations.

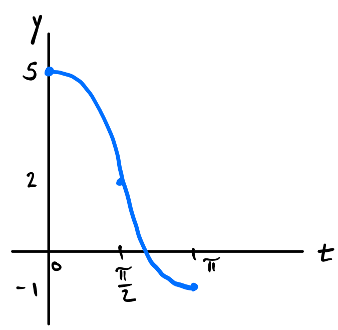
(a) (6 points) Halfway around clockwise, starting at $(-1, 5)$.

(b) (6 points) Twice around counterclockwise, starting at $(-4, 2)$.

(a)



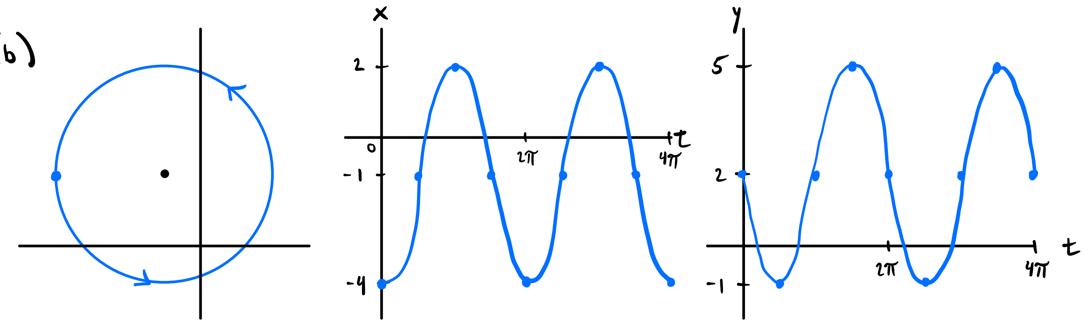




$x = -1 + 3 \sin t$

$y = 2 + 3 \cos t$, $0 \le t \le \pi$

(b)



$$\boxed{\begin{aligned} x &= -1 - 3\cos t \\ y &= 2 - 3\sin t\,, \quad 0 \le t \le 4\pi \end{aligned}}$$

6. Consider the following two polar curves.

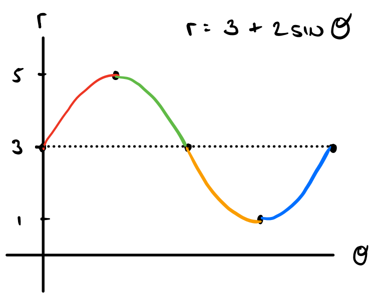
$$r = 3 + 2\sin\theta \tag{1}$$

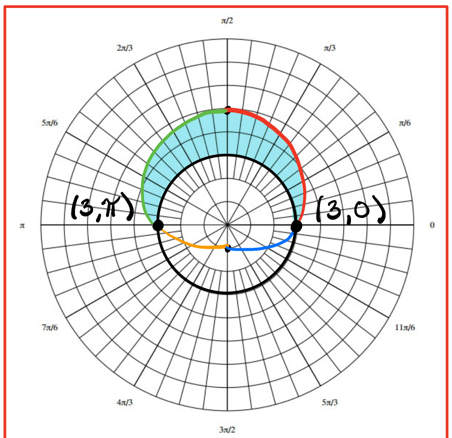
$$r = 3 \tag{2}$$

(a) (10 points) Sketch both polar curves on the axes below and give polar coordinates for all points of intersection.

(b) (14 points) Find the area of the region inside (1) and outside (2).

(a)





(b) Area $A = \displaystyle\int_0^{\pi} \frac{1}{2}(3+2\sin\theta)^2 - \frac{1}{2}(3)^2 \, d\theta$

or, by symmetry, $\displaystyle\int_0^{\pi/2} (3+2\sin\theta)^2 - 3^2 \, d\theta$

$= \displaystyle\int_0^{\pi/2} 12\sin\theta + 4\sin^2\theta \, d\theta = \int_0^{\pi/2} 12\sin\theta + 2(1-\cos 2\theta)\, d\theta$

$= \Big[-12\cos\theta + 2\theta - \sin 2\theta\Big]_0^{\pi/2} = (0+\pi-0) - (-12+0-0) = \boxed{12+\pi}$