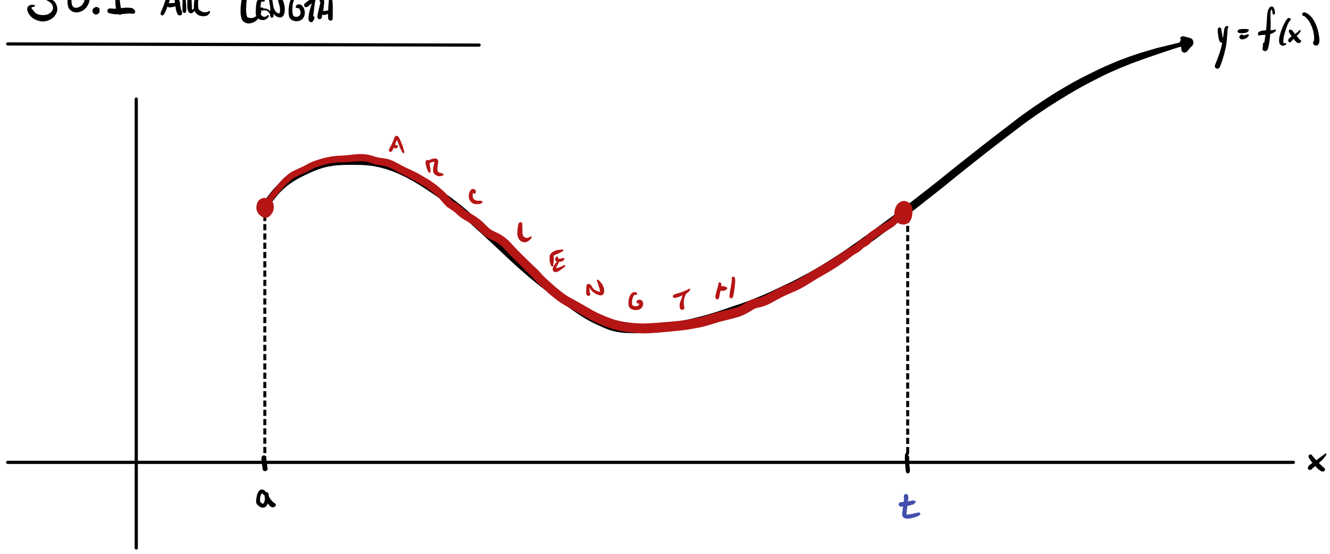
§8.1 Arc Length



Arclength of $y = f(x)$ from a to t $= s(t) = \int_a^t \sqrt{f'(x)^2 + 1}\, dx$

As a func. of x

$$\boxed{s(x) = \int_a^x \sqrt{f'(t)^2 + 1}\, dt}$$

Fundamental Thm of Calculus.

$$\text{FTC} \Rightarrow \frac{ds}{dx} = \sqrt{f'(x)^2 + 1} = \sqrt{\left(\frac{dy}{dx}\right)^2 + 1}, \quad y = f(x)$$

$$ds = \sqrt{\left(\frac{dy}{dx}\right)^2 + 1}\, dx = \sqrt{\left[\left(\frac{dy}{dx}\right)^2 + 1\right] dx^2}$$

$$ds = \sqrt{dy^2 + dx^2}$$

$$ds^2 = dx^2 + dy^2$$

In multivar. calc. this will reappear.

Foreshadowing: Parameterized curves.

ex. FIND THE ARCLENGTH FUNCTION FOR THE CURVE

$y = \underbrace{\sin^{-1}x + \sqrt{1-x^2}}_{f(x)}$ WITH STARTING POINT $(0,1)$

$$s(x) = \int_0^x \sqrt{f'(t)^2 + 1}\, dt$$

$$f(t) = \sin^{-1}t + \sqrt{1-t^2}$$

$$f'(t) = \frac{1}{\sqrt{1-t^2}} + \frac{-t}{\sqrt{1-t^2}} = \frac{1-t}{\sqrt{1-t^2}} \qquad \frac{(1-t)^2}{1-t^2} = \frac{(1-t)\cancel{(1-t)}}{(1+t)\cancel{(1-t)}}$$

$$s(x) = \int_0^x \sqrt{\left(\frac{1-t}{\sqrt{1-t^2}}\right)^2 + 1}\, dt = \int_0^x \sqrt{\frac{(1-t)^2}{1-t^2} + 1}\, dt$$

$$= \int_0^x \sqrt{\frac{1-t}{1+t} + 1}\, dt = \int_0^x \sqrt{\frac{(1-t)+(1+t)}{1+t}}\, dt$$

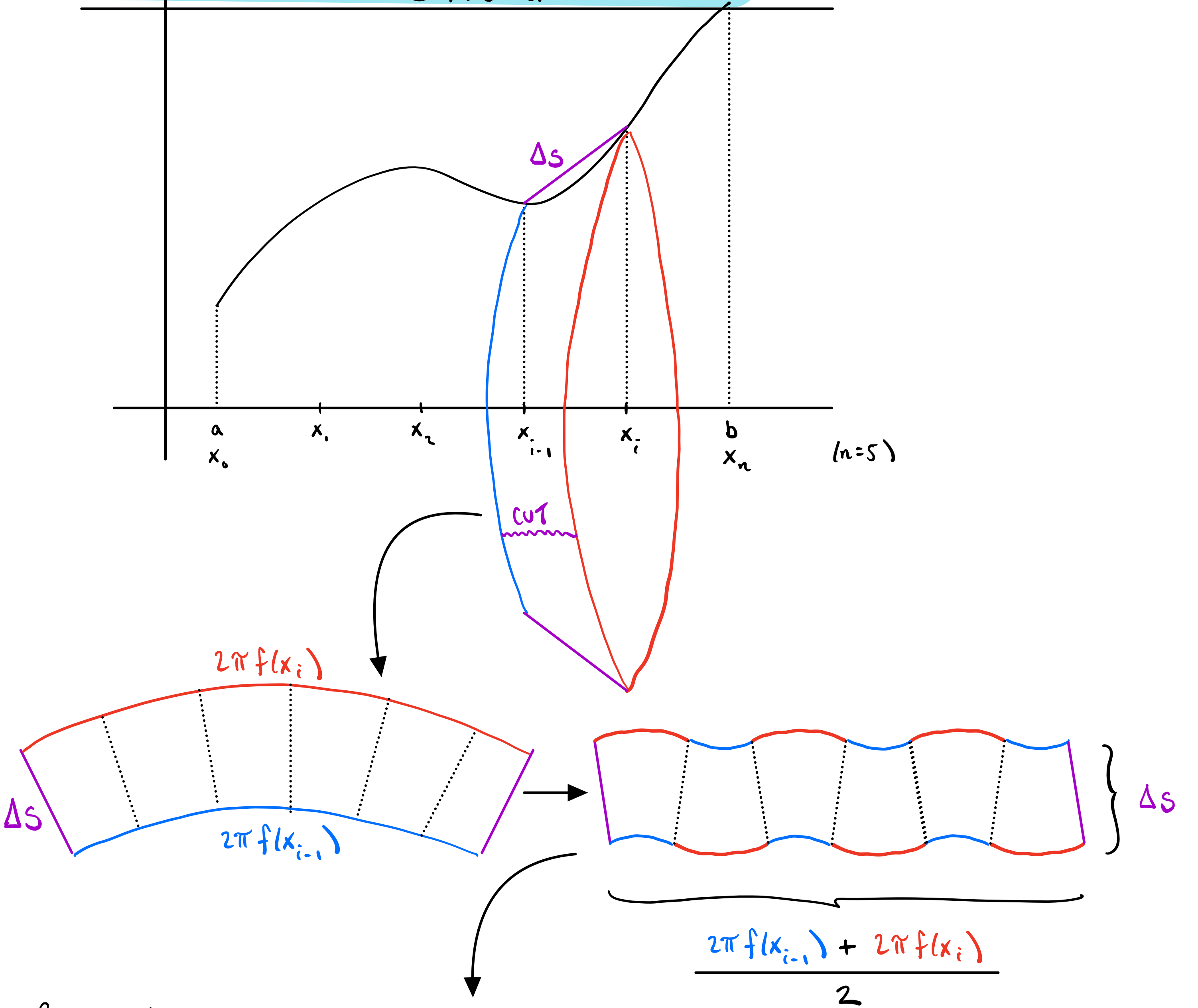
$$= \int_0^x \sqrt{\frac{2}{1+t}}\, dt = \sqrt{2}\int_0^x (1+t)^{-1/2}\, dt = 2\sqrt{2}\,(1+t)^{1/2}\Big|_0^x$$

$$\boxed{s(x) = 2\sqrt{2}\left[(1+x)^{1/2} - 1\right]}$$

NOTE: THE ARCLENGTH OF THE PORTION OF THE CURVE $y = \sin^{-1}x + \sqrt{1-x^2}$ BETWEEN POINTS WITH X-COORDINATES a & b IS $s(b) - s(a)$.

THE ARCLENGTH OF $y = f(x)$ BETWEEN $x=a$ & $x=b$ IS $\displaystyle\int_a^b \sqrt{f'(x)^2 + 1}\, dx$

§ 8.2 Area of a Surface of Revolution



Surface Area

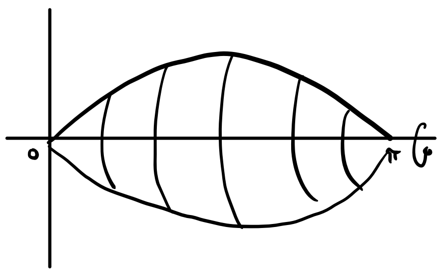
$$A \approx \sum_{i=1}^{n} \overbrace{\frac{2\pi f(x_{i-1}) + 2\pi f(x_i)}{2}} \cdot \Delta s = \sum_{i=1}^{n} 2\pi \left(\underbrace{\frac{f(x_{i-1}) + f(x_i)}{2}} \right) \Delta s$$

Note: By intermediate value thm, $\exists\, \bar{x}_i \in (x_{i-1}, x_i)$ s.t. $f(\bar{x}_i) = \big(f(x_{i-1}) + f(x_i)\big)/2$.

$$A \approx \sum_{i=1}^{n} 2\pi f(\bar{x}_i)\, \Delta s = 2\pi f(\bar{x}_i) \sqrt{1 + f'(x_i^*)^2}\, \Delta x$$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi f(\bar{x}_i) \sqrt{1 + f'(x_i^*)^2}\, \Delta x = \boxed{\int_a^b 2\pi f(x) \sqrt{1 + f'(x)^2}\, dx}$$

ex. FIND THE SURFACE AREA S OF THE SOLID OBTAINED BY ROTATING THE CURVE $y = \sin x$, $0 \le x \le \pi$ ABOUT THE X-AXIS.



$$\boxed{S = 2\pi \int_a^b f(x)\sqrt{f'(x)^2 + 1}\, dx}$$

$$S = 2\pi \int_0^{\pi} \sin x \sqrt{\cos^2 x + 1}\, dx$$

LET $u = \cos x\, dx$
$du = -\sin x\, dx$

$$\rightsquigarrow -2\pi \int_1^{-1} \sqrt{u^2 + 1}\, du$$

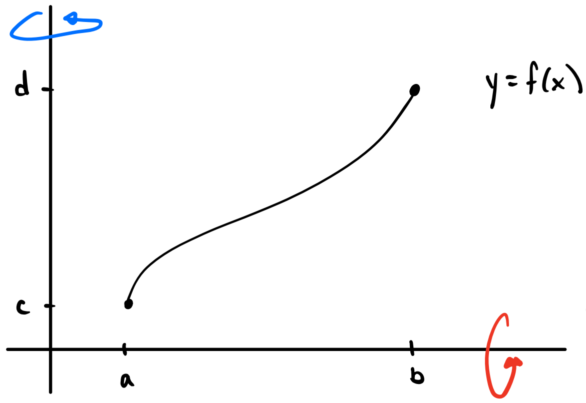
(SWITCH BOUNDS) $2\pi \int_{-1}^{1} \sqrt{u^2+1}\, du$

$u = \tan\theta$
$du = \sec^2\theta\, d\theta$

$$= \underbrace{2\pi \int_{-\pi/4}^{\pi/4} \sec^3\theta\, d\theta}_{\text{INTEGRATING AN EVEN FUNCTION OVER A SYMMETRIC INTERVAL}} = 2\pi \cdot 2 \int_0^{\pi/4} \sec^3\theta\, d\theta = 4\pi \cdot \frac{1}{2}\Big[\sec x \tan x + \ln|\sec x + \tan x|\Big]_0^{\pi/4}$$

$$= 2\pi\left(\sqrt{2} + \ln|\sqrt{2} + 1|\right) \approx 14.4236$$

ALL THE WAYS TO COMPUTE SURFACE AREAS OF SOLIDS OF REVOLUTION



$y = f(x) \iff x = g(y)$

$$ds = \begin{cases} \sqrt{\left(\frac{dy}{dx}\right)^2 + 1}\, dx = \sqrt{f'(x)^2 + 1}\, dx \\ \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy = \sqrt{g'(y)^2 + 1}\, dy \end{cases}$$

(EQUIVALENT CALCULATIONS)

ROTATION ABOUT X-AXIS: $S = \int 2\pi y\, ds = \int_c^d 2\pi y \sqrt{g'(y)^2 + 1}\, dy$ (1)

$= \int_a^b 2\pi f(x) \sqrt{f'(x)^2 + 1}\, dx$ (2)

ROTATION ABOUT y-AXIS: $S = \int 2\pi x\, ds = \int_a^b 2\pi x \sqrt{f'(x)^2 + 1}\, dx$ (3)

$= \int_c^d 2\pi g(y) \sqrt{g'(y)^2 + 1}\, dy$ (4)

7–14 Find the exact area of the surface obtained by rotating the curve about the x-axis.

7. $y = x^3$, $0 \leqslant x \leqslant 2$

8. $y = \sqrt{5 - x}$, $3 \leqslant x \leqslant 5$

9. $y^2 = x + 1$, $0 \leqslant x \leqslant 3$

10. $y = \sqrt{1 + e^x}$, $0 \leqslant x \leqslant 1$

11. $y = \cos(\frac{1}{2}x)$, $0 \leqslant x \leqslant \pi$

12. $y = \dfrac{x^3}{6} + \dfrac{1}{2x}$, $\frac{1}{2} \leqslant x \leqslant 1$

13. $x = \frac{1}{3}(y^2 + 2)^{3/2}$, $1 \leqslant y \leqslant 2$

14. $x = 1 + 2y^2$, $1 \leqslant y \leqslant 2$

15–18 The given curve is rotated about the y-axis. Find the area of the resulting surface.

15. $y = \frac{1}{3}x^{3/2}$, $0 \leqslant x \leqslant 12$

16. $x^{2/3} + y^{2/3} = 1$, $0 \leqslant y \leqslant 1$

17. $x = \sqrt{a^2 - y^2}$, $0 \leqslant y \leqslant a/2$

18. $y = \frac{1}{4}x^2 - \frac{1}{2}\ln x$, $1 \leqslant x \leqslant 2$

7.

$$\int_0^2 2\pi x^3 \sqrt{1+(3x^2)^2}\, dx = 2\pi \int_0^2 x^3 \sqrt{1+9x^4}\, dx$$

$$u = 1+9x^4$$

$$\frac{1}{36}\, du = x^3\, dx \quad \rightsquigarrow \quad \frac{2\pi}{36} \int_0^{145} \sqrt{u}\, du$$

$$= \frac{\pi}{18} \cdot \frac{2}{3} u^{3/2} \Big|_0^{145} = \frac{\pi}{27} \cdot 145^{3/2} \approx 203.16$$

18.

$$\text{AREA} = \int_1^2 2\pi x \sqrt{1+\left(\frac{x}{2}-\frac{1}{2x}\right)^2}\, dx$$

$$= 2\pi \int_1^2 x \sqrt{\frac{x^2}{4}+\frac{1}{2}+\frac{1}{4x^2}}\, dx$$

$$= 2\pi \int_1^2 x \sqrt{\left(\frac{x}{2}+\frac{1}{2x}\right)^2}\, dx = 2\pi \int_1^2 x\left(\frac{x}{2}+\frac{1}{2x}\right) dx$$

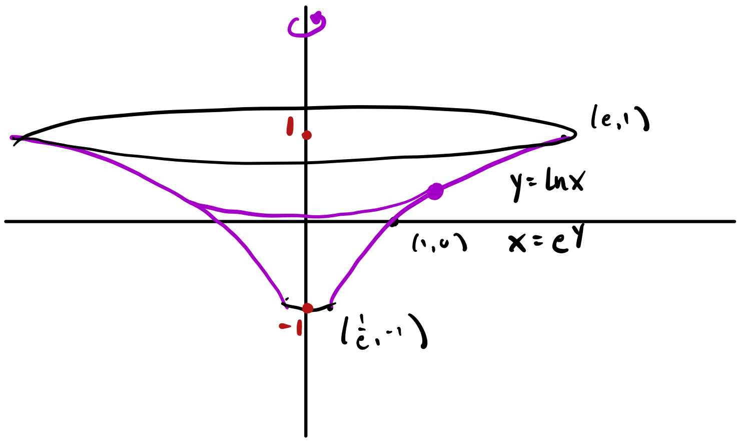
$$= \pi \int_1^2 x^2+1\, dx = \pi\left[\frac{1}{3}x^3 + x\right]_1^2$$

$$= \pi\left[\left(\frac{8}{3}+2\right)-\left(\frac{1}{3}+1\right)\right] = \frac{10\pi}{3}$$

ex. SET UP TWO INTEGRALS FOR SURFACE AREA S OF THE SOLID OBTAINED BY ROTATING $y = \ln x$, $\frac{1}{e} \le x \le e$ AROUND THE y-AXIS.

(a) ONE WITH RESPECT TO x

(b) ONE WITH RESPECT TO y



(a) $A = \int_{1/e}^{e} 2\pi x \sqrt{1 + \left(\frac{1}{x}\right)^2}\, dx$

(b) $A = \int_{-1}^{1} 2\pi e^y \sqrt{1 + (e^y)^2}\, dy$

ex. IF THE INFINITE CURVE $y = e^{-9x}$, $x \ge 0$ IS ROTATED ABOUT THE x-AXIS, FIND THE AREA OF THE RESULTING SURFACE.

LET $f(x) = e^{-9x}$

$f'(x) = -9e^{-9x}$

$$A = \int_0^\infty 2\pi f(x)\sqrt{1 + f'(x)^2}\, dx = 2\pi \int_0^\infty e^{-9x} \sqrt{1 + (-9e^{-9x})^2}\, dx$$

LET $u = -9e^{-9x}$ $\longrightarrow$ WHEN $x = 0$, $u = -9e^0 = -9$

$du = 81e^{-9x}\, dx \Rightarrow \frac{1}{81}\, du = e^{-9x}\, dx$ WHEN $x \to \infty$, $u \to 0$

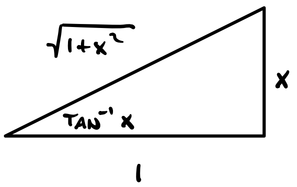
$$\sim \frac{2\pi}{81} \int_{-9}^{0} \sqrt{1 + u^2}\, du$$

LET $u = \tan\theta$, $\sqrt{1+u^2} = \sec\theta$ $\quad \theta = \tan^{-1} u$ $\quad$ WHEN $u = -9$, $\theta = \tan^{-1}(-9)$

$du = \sec^2\theta$ $\quad$ WHEN $u = 0$, $\theta = 0$

$$\rightsquigarrow \frac{2\pi}{81}\int_{\tan^{-1}(-9)}^{0} \sec^3\theta \, d\theta = \frac{2\pi}{81}\cdot\frac{1}{2}\Big[\sec\theta\tan\theta + \ln\left|\sec\theta + \tan\theta\right|\Big]_{\tan^{-1}(-9)}^{0}$$

$$= \frac{\pi}{81}\Big[\left(\sec 0 \tan 0 + \ln\left|\sec 0 + \tan 0\right|\right) - \Big(\sec(\tan^{-1}(-9))\tan(\tan^{-1}(-9))$$
$$+ \ln\left|\sec(\tan^{-1}(-9)) + \tan(\tan^{-1}(-9))\right|\Big)\Big]$$



$$\sec(\tan^{-1}(x)) = \sqrt{1+x^2}$$
$$\Rightarrow \sec(\tan^{-1}(-9)) = \sqrt{82}, \quad \tan(\tan^{-1}(-9)) = -9$$

$$= \frac{\pi}{81}\Big[\big(0 + \underbrace{\ln|1|}_{0}\big) - \big(\sqrt{82}(-9) + \ln\left|\sqrt{82} + (-9)\right|\big)\Big]$$

$$= \boxed{\frac{\pi}{81}\Big[9\sqrt{82} - \ln\left|\sqrt{82} - 9\right|\Big]}$$

EQUIVALENTLY

$$\sqrt{82} - 9 = \frac{(\sqrt{82}-9)(\sqrt{82}+9)}{(\sqrt{82}+9)} = \frac{82-81}{(\sqrt{82}+9)} = \frac{1}{\sqrt{82}+9}$$

$$\Rightarrow -\ln\left|\sqrt{82}-9\right| = -\ln\left|\frac{1}{\sqrt{82}+9}\right| = \ln\left|\sqrt{82}+9\right|$$

$$= \boxed{\frac{\pi}{81}\left(9\sqrt{82} + \ln(\sqrt{82}+9)\right)}$$

integral from 0 to infinity of 2pi*e^(-9x)sqrt(1+(-9e^(-9x))^2)dx

Definite integral

$$\int_0^\infty 2\pi e^{-9x}\sqrt{1+(-9e^{-9x})^2}\,dx = \frac{1}{81}\pi(9\sqrt{82} + \sinh^{-1}(9)) \approx 3.2731$$

More digits | Step-by-step solution

Related Wolfram|Alpha queries

3	$\sinh^{-1}x = \ln\left(x + \sqrt{x^2+1}\right)$	$x \in \mathbb{R}$
4	$\cosh^{-1}x = \ln\left(x + \sqrt{x^2-1}\right)$	$x \geqslant 1$
5	$\tanh^{-1}x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$	$-1 < x < 1$

$$\Rightarrow \sinh^{-1}(9) = \ln(9+\sqrt{82}) \checkmark$$