§11.8 Power Series

N^{TH} Degree Polynomial :

$$P(x) = \sum_{n=0}^{N} c_n x^n = c_0 + c_1 x + c_2 x^2 + \dots + c_N x^N$$

The c_n are coefficients
c_0 is the constant term.

" ∞ degree polynomial" ?

Def: A Power Series about (centered at) $x=0$ is a series of the form

$$f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \dots$$

A Power Series about $x=a$ is a series of the form

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \dots$$

We call a the center & the constants c_0, c_1, c_2 are called the coefficients.

ex. Power Series about $x=0$ with all coefficients equal to 1 : $c_n = 1 \ \forall n$.

$$f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \dots$$

$f(-1) = 1 - 1 + 1 - 1 + 1 - 1 + \dots$ = diverges.

$f(0) = 1 + 0 + 0 + 0 + \dots = 1.$

$f(\frac{1}{2}) = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots = 2$ (Geometric Series)

RECALL <u>GEOMETRIC SERIES:</u>

$$\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \ldots$$

CONVERGES TO $\frac{a}{1-r}$ IF $|r| < 1$

DIVERGES IF $|r| \geq 1$.

$$f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \ldots$$

CONVERGES TO $\frac{1}{1-x}$ IF $|x| < 1$

DIVERGES IF $|x| \geq 1$.

IN OTHER WORDS,

$$\frac{1}{1-x} = 1 + x^2 + x^3 + x^4 + \ldots \quad \text{IF } -1 < x < 1.$$

IN THIS CASE, WE HAVE 2 WAYS OF REPRESENTING THE SAME FUNCTION!

BUT IN GENERAL, THESE <u>ARE</u> DIFFERENT FUNCTIONS BECAUSE THEIR DOMAINS ARE DIFFERENT.

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ex. FOR WHAT VALUES OF x DOES POWER SERIES $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ CONVERGE?

$(0! = 1)$

$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, WHAT'S THE DOMAIN OF f?

$f(0) = 0$, 0 IS IN $\mathrm{Dom}(f)$

CONVERGES AT $x = 0$.

RATIO TEST: $\rho(x) = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{x^{\cancel{n+1}}}{(n+1)\cancel{!}} \cdot \frac{\cancel{n!}}{\cancel{x^n}} \right|$

$\rho(x) = \underbrace{\lim_{n \to \infty} \left| \frac{x}{n+1} \right|}_{x \text{ IS FIXED.}} = 0$ FOR ALL x.

$\therefore$ SERIES CONVERGES FOR ALL x.

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ex. FOR WHAT VALUES OF x DOES POWER SERIES ABOUT $x = 1$ CONVERGE?

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{\sqrt{n}}$$ (OBVIOUSLY CONVERGES WHEN $x = 1$)

NOTE: SINCE WE DON'T KNOW THE VALUE OF x, WE DON'T KNOW IF THE SERIES IS ALTERNATING OR NOT.

SO WE TEST FOR ABSOLUTE CONVERGENCE.

- INTEGRAL TEST
- DIRECT COMPARISON TEST
- LIMIT COMPARISON TEST
- RATIO TEST
- ROOT TEST

RATIO TEST: $\rho(x) = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(x-1)^{\cancel{n+1}}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{\cancel{(x-1)^n}} \right|$

$= \lim_{n \to \infty} \sqrt{\frac{n}{n+1}} \cdot \underbrace{\lim_{n \to \infty} |x-1|}_{x \text{ IS FIXED}} = |x-1|$

SERIES CONVERGES ABSOLUTELY IF $\rho(x) < 1$

SERIES DIVERGES IF $\rho(x) > 1$.

$\sum_{n=1}^{\infty} \frac{(x-1)^n}{\sqrt{n}}$ CONVERGES IF $|x-1| < 1$

$-1 < x-1 < 1$

$0 < x < 2$

DIVERGES IF $|x-1| > 1$

$x < 0$ OR $x > 2$

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{\sqrt{n}} = \begin{cases} \text{DIVERGES IF } x < 0 \\ \text{CONVERGES IF } x = 0 \\ \text{CONVERGES IF } 0 < x < 2 \\ \text{DIVERGES IF } x = 2 \\ \text{DIVERGES IF } 2 < x \end{cases}$$

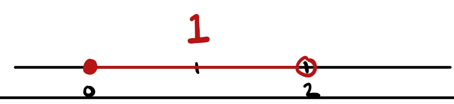
$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ ALTERNATING, TERMS $\to 0$

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ DIVERGES, P-SERIES TEST $p = \frac{1}{2} < 1$

$\frac{1}{n^{1/2}}$

$$\boxed{\sum_{n=1}^{\infty} \frac{(x-1)^n}{\sqrt{n}} \text{ CONVERGES FOR } x \text{ IN } [0, 2)}$$

CENTER



Convergence Theorem for Power Series

IF THE POWER SERIES $\sum_{n=0}^{\infty} c_n x^n$ ($\sum_{n=0}^{\infty} c_n (x-a)^n$) CONVERGES AT $x = b \neq 0$

THEN IT CONVERGES ABSOLUTELY FOR ALL x SUCH THAT $|x| < |b|$.

$|x-a| < |b-a|$

IF THE POWER SERIES $\sum_{n=0}^{\infty} c_n x^n$ ($\sum_{n=0}^{\infty} c_n (x-a)^n$) DIVERGES AT $x = d$,

THEN IT DIVERGES FOR ALL x SUCH THAT $|x| > |d|$.

$|x-a| > |d-a|$

PROOF: 1 SUPPOSE $\sum_{n=0}^{\infty} c_n x^n$ CONVERGES AT $x = b$.

$\Rightarrow \sum_{n=0}^{\infty} c_n b^n$ CONVERGES $\Rightarrow \lim_{n\to\infty} c_n b^n = 0$ (DIV. TEST)

THERE EXISTS AN INTEGER N SUCH THAT FOR ALL $n \geq N$ WE HAVE

$$|c_n b^n| < 1 \quad \Rightarrow \quad \boxed{|c_n| < \frac{1}{|b|^n}} \; (*) .$$

NOW LET $|x| < |b|$, SO $\left|\frac{x}{b}\right| < 1$

THEN $(*) \Rightarrow |x|^n |c_n| < \frac{1}{|b|^n} |x|^n$

$$|c_n x^n| < \left|\frac{x}{b}\right|^n$$

$\sum \left|\frac{x}{b}\right|^n$ CONVERGES $\left(\text{GEOMETRIC SERIES WITH } r = \left|\frac{x}{b}\right| < 1\right)$

$\therefore \sum |c_n x^n|$ CONVERGES (DIRECT COMP. TEST)

$\therefore \sum c_n x^n$ CONVERGES ABSOLUTELY. ✓

2 SUPPOSE $\sum_{n=0}^{\infty} c_n x^n$ DIVERGES AT $x = d$.

WE SHOW THAT THE SERIES CANNOT CONVERGE AT ANY x SUCH THAT $|x| > |d|$.

(PROOF BY CONTRADICTION)

ASSUME THE SERIES CONVERGES AT SOME x WITH $|x| > |d|$.

THEN PART 1 IMPLIES THAT THE SERIES CONVERGES AT $x = d$.

THIS IS A CONTRADICTION!

THUS OUR ASSUMPTION MUST BE FALSE.

$\therefore$ THE SERIES DIVERGES AT ALL x WITH $|x| > |d|$. ∎

ex. FOR WHAT VALUES OF x DOES $\displaystyle\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n}$ CONVERGE ?

ROOT TEST: $\displaystyle \rho(x) = \lim_{n\to\infty} \left| \frac{(3x-2)^n}{n} \right|^{1/n} = \lim_{n\to\infty} \frac{|3x-2|}{n^{1/n}} \longrightarrow 1$

$$\rho(x) = |3x-2|$$

SERIES CONVERGES IF $\rho(x) < 1$

$$|3x-2| < 1$$

$$-1 < 3x-2 < 1$$
$$1 < 3x < 3$$

$$\frac{1}{3} < x < 1$$

DIVERGES IF $\rho(x) > 1$

$$x < \frac{1}{3} \quad \text{OR} \quad 1 < x$$

IF $x = \frac{1}{3}$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ CONVERGES ✓

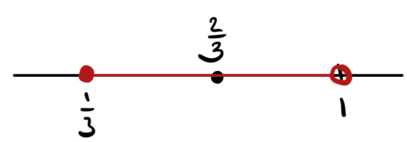
ALT. HARMONIC

IF $x = 1$: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ DIVERGES ✗

HARMONIC

$\displaystyle\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n}$ CONVERGES

FOR ALL x IN $\left[\frac{1}{3}, 1\right)$



Note:

$$\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n} = \sum_{n=1}^{\infty} \frac{\left(3\left(x-\frac{2}{3}\right)\right)^n}{n}$$

$$= \sum_{n=1}^{\infty} \frac{3^n}{n} \left(x - \frac{2}{3}\right)^n$$

POWER SERIES CENTERED ABOUT $x = \frac{2}{3}$.

CONVERGENCE THEOREM FOR POWER SERIES

$x = a + b$

IF THE POWER SERIES $\sum_{n=0}^{\infty} c_n (x-a)^n$ CONVERGES AT $x - a = b \neq 0$

THEN IT CONVERGES ABSOLUTELY FOR ALL x SUCH THAT $|x-a| < |b|$.

IF THE POWER SERIES $\sum_{n=0}^{\infty} c_n (x-a)^n$ DIVERGES AT $x - a = d$,

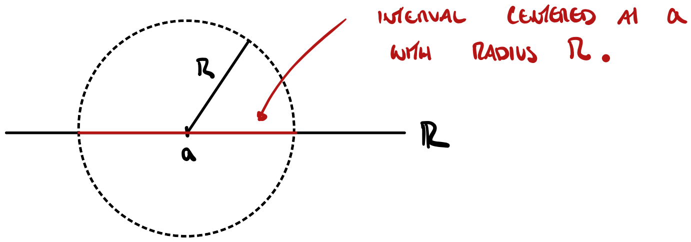
$x = a + d$

THEN IT DIVERGES FOR ALL x SUCH THAT $|x-a| > |d|$.

(REPLACE x WITH $x-a$)

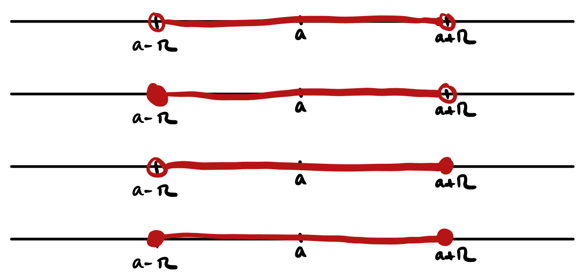
RADIUS OF CONVERGENCE FOR POWER SERIES

GIVEN A POWER SERIES $\sum_{n=0}^{\infty} c_n (x-a)^n$, THE SET OF ALL x SUCH THAT THE SERIES CONVERGES IS AN INTERVAL CENTERED AT a WITH RADIUS R, RADIUS OF CONVERGENCE



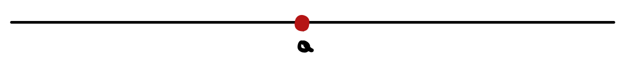
Three Cases: (1) $0 < R < \infty$: Series converges for $|x-a| < R$
diverges for $|x-a| > R$

At endpoints $x = a \pm R$ we must check for convergence explicitly.



(2) $R = \infty$: Series converges for all x.

(3) $R = 0$: The series converges at $x=a$ only.



To find interval of convergence:

1. Ratio/root test to find radius of convergence.

   This gives the largest open interval on which the series converges absolutely.

2. If $0 < R < \infty$, test endpoints $x = a \pm R$ separately for convergence/divergence (DCT, LCT, Int Test, Alt Series Test)

   Note: Convergence at endpoints may be conditional.

3. If $R < \infty$, then series diverges for $|x-a| > R$.

ex. FIND RADIUS OF CONVERGENCE AND ALL VALUES x SUCH THAT POWER SERIES CONVERGES

$$\sum_{n=0}^{\infty} \frac{n(x+3)^n}{5^n}$$

ROOT TEST : $\rho(x) = \lim_{n\to\infty} \left| \frac{n(x+3)^n}{5^n} \right|^{1/n} = \lim_{n\to\infty} n^{1/n} \left| \frac{x+3}{5} \right|$

$$= \underbrace{\lim_{n\to\infty} n^{1/n}}_{\substack{1 \\ (\text{L'HO. RULE})}} \cdot \lim_{n\to\infty} \left| \frac{x+3}{5} \right| = \left| \frac{x+3}{5} \right|$$

SERIES CONVERGES ABSOLUTELY IF $\rho(x) < 1$

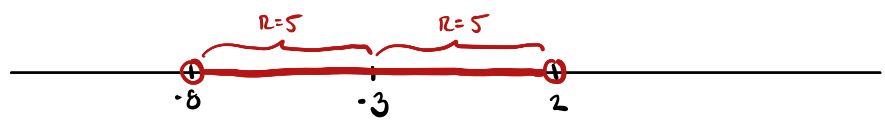
$\Rightarrow \left| \frac{x+3}{5} \right| < 1 \Rightarrow |x+3| < 5$

RADIUS OF CONVERGENCE $R = 5$

$|x-(-3)| < 5$ DIST. TO -3 IS < 5

$-5 < x+3 < 5$

$-8 < x < 2$



WHAT ABOUT ENDPOINTS?

IF $x = -8$: $\sum_{n=0}^{\infty} \frac{n(-8+3)^n}{5^n} = \sum_{n=0}^{\infty} \frac{n(-5)^n}{5^n} = \sum_{n=0}^{\infty} (-1)^n n$

DIVERGES SINCE $\lim_{n\to\infty} (-1)^n n \neq 0$ (DIV. TEST)

IF $x = 2$: $\sum_{n=0}^{\infty} \frac{n(2+3)^n}{5^n} = \sum_{n=0}^{\infty} n$ DIVERGES ALSO.

RADIUS OF CONVERGENCE $R = 5$,

INT. OF CONVERGENCE $(-8, 2)$.

ex. FIND RADIUS OF CONVERGENCE AND ALL VALUES x SUCH THAT POWER SERIES CONVERGES

$$\sum_{n=1}^{\infty} n^n x^n$$

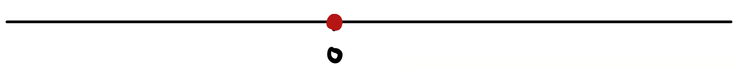
ROOT TEST: $\rho(x) = \lim_{n\to\infty} \left| n^n x^n \right|^{1/n} = \lim_{n\to\infty} |nx| = \begin{cases} 0 & \text{IF } x = 0 \\ \infty & \text{IF } x \neq 0 \end{cases}$

$\therefore$ $\rho(x) < 1$ IF $x = 0$

$\rho(x) > 1$ IF $x \neq 0$

RADIUS OF CONVERGENCE 0.

SERIES CONVERGES AT $x = 0$ ONLY.



3–28 Find the radius of convergence and interval of convergence of the series.

3. $\displaystyle\sum_{n=1}^{\infty} (-1)^n n x^n$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\sqrt[3]{n}}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{2n-1}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n^2}$

7. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

8. $\displaystyle\sum_{n=1}^{\infty} n^n x^n$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^4 4^n}$

10. $\displaystyle\sum_{n=1}^{\infty} 2^n n^2 x^n$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 4^n}{\sqrt{n}} x^n$

12. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n 5^n} x^n$

13. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n(n^2+1)} x^n$

14. $\displaystyle\sum_{n=1}^{\infty} \frac{x^{2n}}{n!}$

15. $\displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2+1}$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)2^n} (x-1)^n$

17. $\displaystyle\sum_{n=2}^{\infty} \frac{(x+2)^n}{2^n \ln n}$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{8^n} (x+6)^n$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^n}$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n \sqrt{n}}$

21. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{b^n} (x-a)^n, \quad b > 0$

22. $\displaystyle\sum_{n=2}^{\infty} \frac{b^n}{\ln n} (x-a)^n, \quad b > 0$

23. $\displaystyle\sum_{n=1}^{\infty} n!(2x-1)^n$

24. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 x^n}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)}$

25. $\displaystyle\sum_{n=1}^{\infty} \frac{(5x-4)^n}{n^3}$

26. $\displaystyle\sum_{n=2}^{\infty} \frac{x^{2n}}{n(\ln n)^2}$

27. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}$

28. $\displaystyle\sum_{n=1}^{\infty} \frac{n! x^n}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}$