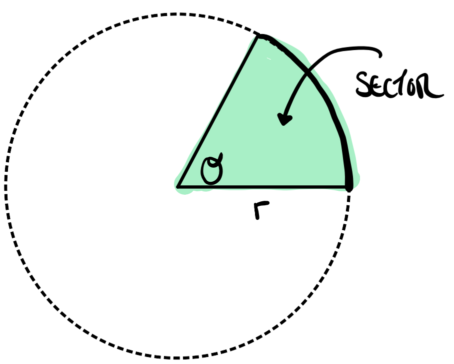
§10.4 Areas & Lengths in Polar Coordinates

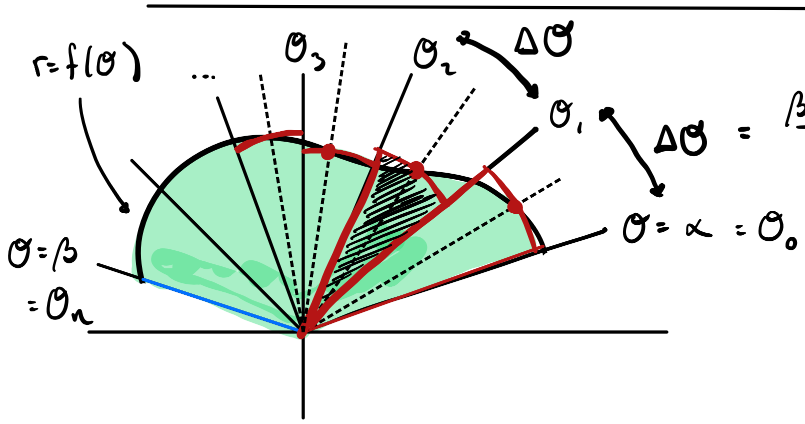
Area of a Sector



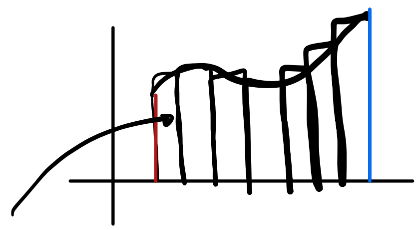
$$\text{Area} = (\text{fraction})\pi r^2 = \left(\frac{\theta}{2\pi}\right)\pi r^2$$

$$\boxed{\text{Area} = \frac{r^2\theta}{2}}$$

Area Bounded by Polar Graphs $r = f(\theta)$, $\theta = \alpha$, $\theta = \beta$



$$\Delta\theta = \frac{\beta - \alpha}{n}$$



$$\int y\,dx = \int f(x)\,dx$$

$y = f(x)$

To find the area bounded by polar curves $r = f(\theta)$, $\theta = \alpha$, $\theta = \beta$

break interval $[\alpha, \beta]$ up into n subintervals.

Since $r = f(\theta)$ is continuous, we can approximate r by a constant on each subinterval $[\theta_{i-1}, \theta_i]$

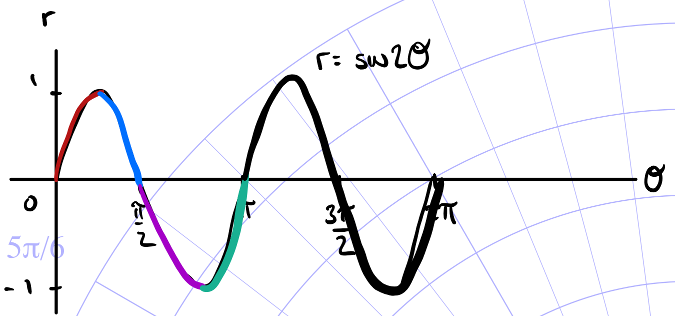
This breaks the region up into n sectors, the area of the i^{th} sector

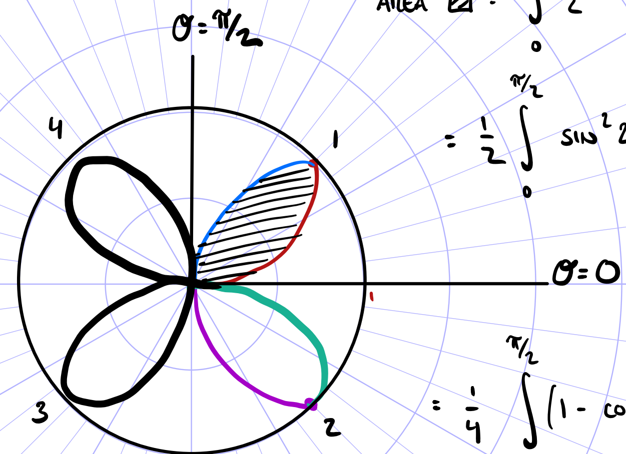
$$A_i = r(\theta_i)^2 \frac{\Delta\theta}{2}$$

$$\text{Total Area} \approx \sum_{i=1}^{n} r(\theta_i)^2 \frac{\Delta\theta}{2}.$$

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} r(\theta_i)^2 \frac{\Delta\theta}{2} = \int_{\alpha}^{\beta} \frac{1}{2} r(\theta)^2\,d\theta$$

ex. FIND THE AREA ENCLOSED BY ONE LOOP/PEDAL OF THE POLAR CURVE $r = \sin 2\theta$.





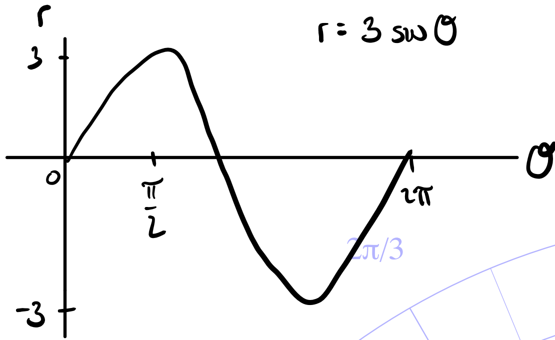
$$\text{AREA} = \int_0^{\pi/2} \frac{1}{2} r^2 \, d\theta$$

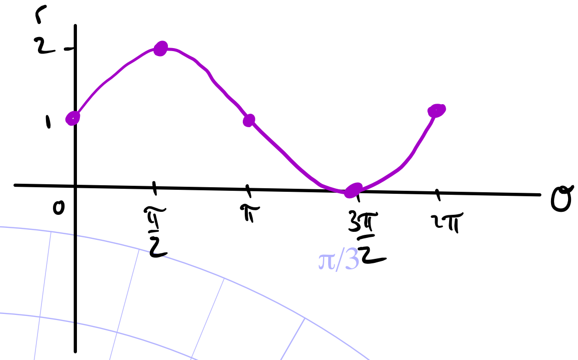
$$= \frac{1}{2} \int_0^{\pi/2} \sin^2 2\theta \, d\theta$$

$$= \frac{1}{4} \int_0^{\pi/2} (1 - \cos 4\theta) \, d\theta$$

$$= \frac{1}{4} \left[\theta - \frac{1}{4} \sin 4\theta \right]_0^{\pi/2} = \boxed{\frac{\pi}{8}}$$

EXAMPLE 2 Find the area of the region that lies inside the circle $r = 3\sin\theta$ and outside the cardioid $r = 1 + \sin\theta$.



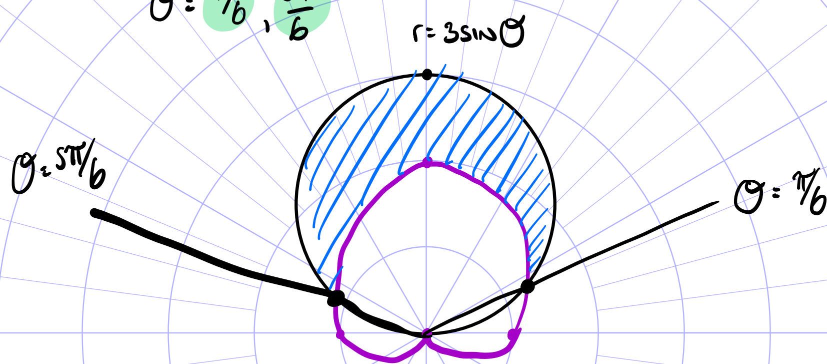


INTERSECTIONS: $3\sin\theta = 1 + \sin\theta$

$2\sin\theta = 1$

$\sin\theta = \frac{1}{2}$

$\theta = \pi/6,\ \frac{5\pi}{6}$



$$\text{AREA} = \int_{\alpha}^{\beta} \frac{1}{2}\left(r_{out}^2 - r_{in}^2\right) d\theta = \frac{1}{2}\int_{\pi/6}^{5\pi/6} (3\sin\theta)^2 - (1+\sin\theta)^2 \, d\theta$$

$$= \int_{\pi/6}^{\pi/2} (3\sin\theta)^2 - (1+\sin\theta)^2 \, d\theta$$

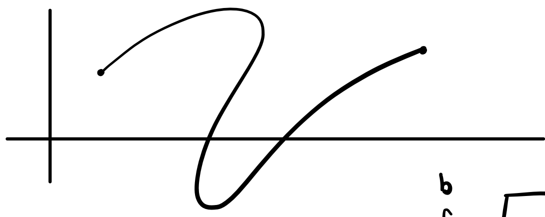
ARC LENGTH

RECALL ARCLENGTH OF PARAMETRIC CURVE :

$x = f(t)$

$y = g(t)$

$a \le t \le b$

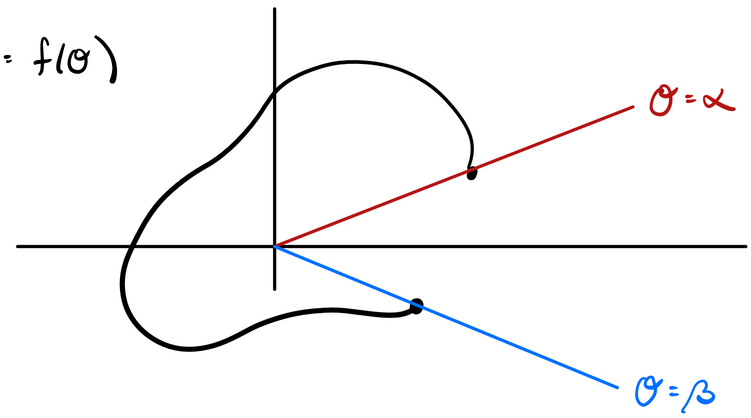


$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$x = r\cos\theta$ WHERE $r = f(\theta)$

$y = r\sin\theta$

$\alpha \le \theta \le \beta$



$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= \int_\alpha^\beta \sqrt{\left(\frac{dr}{d\theta}\cos\theta - r\sin\theta\right)^2 + \left(\frac{dr}{d\theta}\sin\theta + r\cos\theta\right)^2}\, d\theta$$

$$= \int_\alpha^\beta \sqrt{\left(\frac{dr}{d\theta}\right)^2\cos^2\theta - 2r\frac{dr}{d\theta}\sin\theta\cos\theta + r^2\sin^2\theta + \left(\frac{dr}{d\theta}\right)^2\sin^2\theta + 2r\frac{dr}{d\theta}\sin\theta\cos\theta + r^2\cos^2\theta}\, d\theta$$

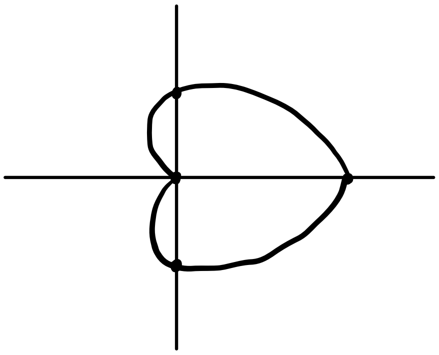
$$= \int_\alpha^\beta \sqrt{\left(\frac{dr}{d\theta}\right)^2\left[\cos^2\theta + \sin^2\theta\right] + r^2\left[\cos^2\theta + \sin^2\theta\right]}\, d\theta$$

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

ex. FIND THE EXACT LENGTH OF THE CARDIOID $r = 1 + \cos\theta$

HINT: $\cos^2(x) = \frac{1}{2}(1 + \cos(2x)) \Rightarrow 2\cos^2(x) = 1 + \cos(2x)$

$\Rightarrow 2\cos^2(\frac{1}{2}\theta) = 1 + \cos(\theta)$



$$L = 2\int_0^{\pi} \sqrt{(1+\cos\theta)^2 + (-\sin\theta)^2}\, d\theta$$

$$= 2\int_0^{\pi} \sqrt{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta}\, d\theta$$

$$= 2\sqrt{2}\int_0^{\pi} \sqrt{1+\cos\theta}\, d\theta = 2\sqrt{2}\int_0^{\pi} \sqrt{2\cos^2(\tfrac{1}{2}\theta)}\, d\theta$$

$$= 4\int_0^{\pi} \underbrace{|\cos(\tfrac{1}{2}\theta)|}_{\text{Pos. } \checkmark}\, d\theta = 8\sin(\tfrac{1}{2}\theta)\Big|_0^{\pi} = 8$$