FINAL EXAM TUESDAY 12/15 1:30 - 3:30 PM (ZOOM/PAPER)
(CUMULATIVE)

§6.4* General Logarithmic & Exponential Functions

Given any $b > 0$, we define the exponential function with base b to be b^x.

Note: We do require $b > 0$.

If $b < 0$ ... say $b = -2$: $f(x) = (-2)^x$

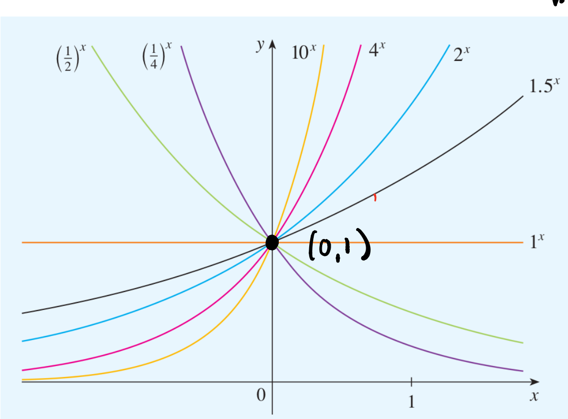
Dom(f) does not include any rational # with even denom.

e.g. $f(\frac{1}{2}) = (-2)^{1/2} = \sqrt{-2}$ ⊗

IMPORTANT: $\boxed{b^x = e^{x \ln b}}$

$$[b]^x = [e^{\ln b}]^x = e^{\ln(b) x} = e^{x \ln b}$$

Graphs $y = b^x$ $1 = b^0$ for all b.



$y = e^{x \ln b}$ if $f(x) = e^x$ & $\ln b = c$

then $y = f(cx)$

$y = f(x) = e^x$

IF $\begin{cases} c < 0 \\ \ln b < 0 \\ b < 1 \end{cases}$ THEN THE GRAPH $y = f(cx)$ IS REFLECTED ACROSS THE y-AXIS.

$y = b^x$

e.g. $y = \left(\frac{1}{a}\right)^x \Rightarrow y = (a^{-1})^x = a^{-x}$

$a > 1$

REFLECTIONS OF EACH OTHER ACROSS y-AXIS.

$y = a^x$

ex. Suppose a population grows exponentially. That is, the population at time t is given by

$$P(t) = Cb^t, \qquad C \text{ \& } b \text{ ARE CONSTANTS}$$

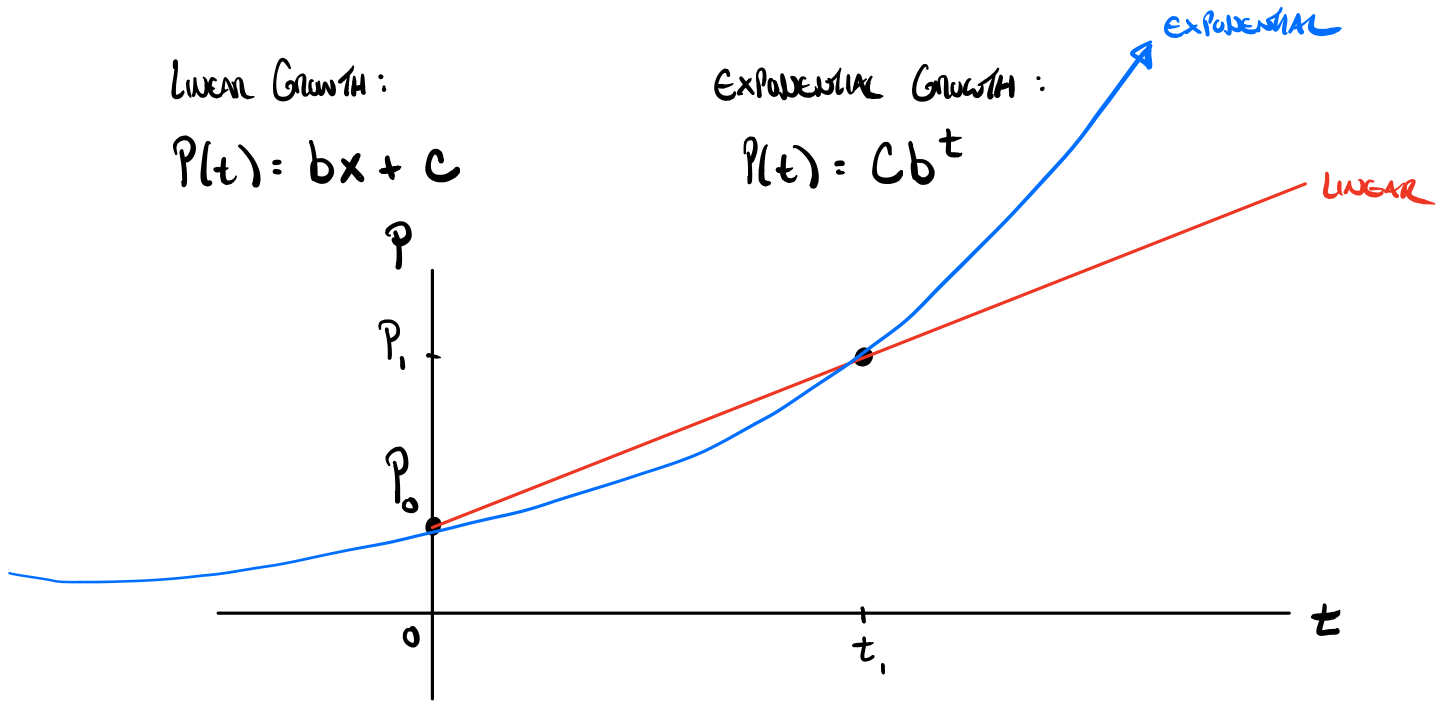
If a population has size P_0 At time t=0 and has size P_1 at time $t=t_1$, find an expression for the population at time t.

LINEAR GROWTH:

$$P(t) = bx + c$$

EXPONENTIAL GROWTH:

$$P(t) = Cb^t$$



GIVEN $P(0) = Cb^0 = P_0$

$$C = P_0$$

$$\Rightarrow \quad P(t) = P_0 b^t$$

CHECK: $P(0) = P_0 b^0$

$= P_0 \times 1$

$= P_0$ ✓

$$P(t_1) = \underbrace{P_0 b^{t_1} = P_1}_{\text{SOLVE FOR } b.}$$

$$b^{t_1} = \frac{P_1}{P_0}$$

$$\left(b^{t_1}\right)^{1/t_1} = \left(\frac{P_1}{P_0}\right)^{1/t_1}$$

$$b = \left(\frac{P_1}{P_0}\right)^{1/t_1}$$

C   b

$$\boxed{P(t) = P_0 \left(\sqrt[t_1]{\frac{P_1}{P_0}}\right)^t}$$

3 **Laws of Exponents** If x and y are real numbers and $a, b > 0$, then

1. $b^{x+y} = b^x b^y$ **2.** $b^{x-y} = \dfrac{b^x}{b^y}$ **3.** $(b^x)^y = b^{xy}$ **4.** $(ab)^x = a^x b^x$

PROOF OF 1: $b^{x+y} = \left(e^{\ln b}\right)^{x+y} = e^{x\ln b + y\ln b}$

$= e^{x\ln b} e^{y\ln b} = \left[e^{\ln b}\right]^x \left[e^{\ln b}\right]^y$

$= b^x b^y$ $\square$

$$\frac{d}{dx}(b^x) = b^x \ln b$$

b = CONSTANT

PROOF: $\dfrac{d}{dx}\left[b^x\right] = \dfrac{d}{dx}\left[e^{x\ln b}\right]$

$= e^{x\ln b}\dfrac{d}{dx}\left[x\ln b\right]$

$= e^{x\ln b}\ln b = b^x \ln b$

$f(x) = b^x \Rightarrow f'(x) = \ln b\ f(x)$

f' IS A MULTIPLE OF f

$\dfrac{d}{dx} e^x = e^x \underbrace{\ln e}_{1} = e^x$

ex. DIFFERENTIATE $f(x) = 3^{\cos(2x)}$

LET $u = \cos(2x) \longrightarrow f(x) = 3^u$

$\dfrac{du}{dx} = -\sin(2x)\,2$

$f'(x) = \dfrac{d}{du}\left[3^u\right]\dfrac{du}{dx}$

$= 3^u \ln 3 \cdot (-2\sin(2x))$

$= 3^{\cos(2x)}\ln(3)(-2\sin(2x))$

ex. DIFFERENTIATE $f(x) = \tan\left(4^{x^2}\right)$

$f'(x) = \sec^2\left(4^{x^2}\right) \cdot 4^{x^2}\ln 4 \cdot 2x$

$= \underbrace{(2\ln 4)}_{\ln 16} \times 4^{x^2}\sec^2\left(4^{x^2}\right)$

Power Func. x^n Variable Base, Constant Exp

Exp. Func. b^x Variable Exp, Constant Base

ex. Differentiate $f(x) = [x]^x$

$\boxed{b^x = e^{x \ln b}}$

$$f(x) = \left[e^{\ln x}\right]^x = e^{x \ln x}$$

$$f'(x) = e^{x\ln x} \cdot \frac{d}{dx}[x \ln x]$$

$$= e^{x\ln x}\left(x \cdot \frac{1}{x} + 1 \ln x\right) = \boxed{x^x(1 + \ln x)}$$

$$\int b^x\,dx = \frac{b^x}{\ln b} + C \qquad b \neq 1$$

Check: $\frac{d}{dx}\left[\frac{1}{\ln b} b^x + c\right]$

$$= \frac{1}{\ln b}\frac{d}{dx} b^x + 0 = \frac{1}{\ln b} b^x \ln b = b^x \checkmark$$

ex. $\int 6^{\sin(3x)} \cos(3x)\,dx$

Let $u = \sin(3x)$

$du = 3\cos(3x)\,dx$

$\frac{1}{3}du = \cos(3x)\,dx$

$$\rightsquigarrow \frac{1}{3}\left[\int 6^u\,du\right] = \frac{1}{3}\left[\frac{6^u}{\ln 6} + C_0\right]$$

$$\frac{6^{\sin(3x)}}{3\ln 6} + C_1$$

The Power Rule If n is any real number and $f(x) = x^n$, then

$$f'(x) = nx^{n-1}$$

PROOF: LET $y = x^n$. FIND y'.

$$\ln(y) = \ln(x^n) = n \ln x$$

$$\frac{1}{y} y' = n \cdot \frac{1}{x}$$

$$y' = n \cdot \frac{y}{x} = n \cdot \frac{x^n}{x} = n x^{n-1} \quad \square$$

$$y = x^{p/q} \qquad p, q \in \mathbb{Z}$$

$$y^q = x^p$$

$$q y^{q-1} y' = p x^{p-1} \qquad \text{IMP. DIFF.}$$

$$y' = \frac{p x^{p-1}}{q y^{q-1}} = \ldots \frac{p}{q} x^{\frac{p}{q}-1}$$

ALGEBRAIC EXERCISE

In general there are four cases for exponents and bases:

Constant base, constant exponent

1. $\frac{d}{dx}(b^n) = 0$ (b and n are constants)

Variable base, constant exponent

2. $\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1} f'(x)$

Constant base, variable exponent

3. $\frac{d}{dx}[b^{g(x)}] = b^{g(x)}(\ln b) g'(x)$

Variable base, variable exponent

4. To find $(d/dx)[f(x)]^{g(x)}$, logarithmic differentiation can be used, as in the next example.

EXAMPLE 4 Differentiate $y = x^{\sqrt{x}}$.

For any $b > 0$, we define the general logarithmic function with base b, $\log_b x$, to be the inverse function of the general exponential function with base b.

That is,

$$\log_b x = y \iff b^y = x$$

"log-base-b" "exponential-base-b"

Note:

(1) $\log_b b = 1 \quad : \quad b^1 = b$

(2) $\log_b 1 = 0 \quad : \quad b^0 = 1$

(3) $\log_e x = \ln x$

Note: $\log_b b^x = x$

$b^{\log_b x} = x \, , \quad x > 0$

ex. $\log_3 27 = 3 \quad : \quad 3^3 = 27$

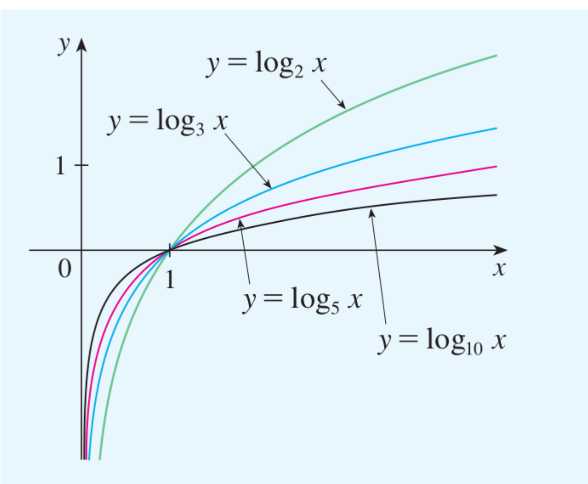
$\log_3 9 = 2 \; : \; 3^2 = 9$

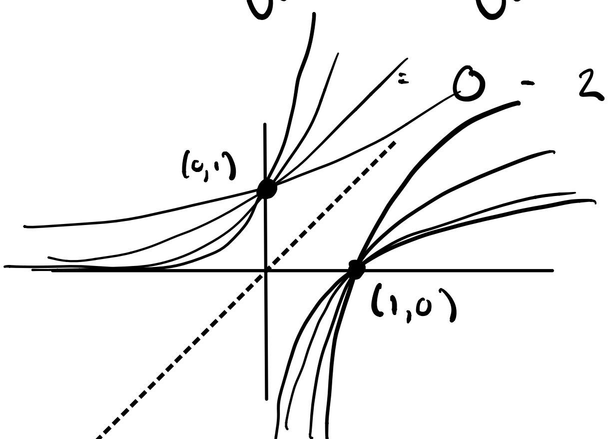
ex. $\log_5 \frac{1}{25} = -2 \quad : \quad 5^{-2} = \frac{1}{25}$

$5^2 = 25$

Alt: $\log_5 \frac{1}{25} = \log_5 1 - \overbrace{\log_5 25}^{2}$

$= 0 - 2 = -2$





Inverse graphs are reflections across $y = x$.

6 Change of Base Formula For any positive number b ($b \neq 1$), we have

$$\log_b x = \frac{\ln x}{\ln b}$$

IMPORTANT

$$\log_7 6 = \frac{\ln(6)}{\ln(7)}$$

PROOF: Let $y = \log_b x$.

$$b^y = x \longrightarrow \ln(b^y) = \ln x$$

$$y \ln b = \ln x \rightarrow y = \frac{\ln x}{\ln b} \quad \square$$

$$\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$$

$$\frac{d}{dx}\left[\frac{\ln x}{\ln b}\right] = \frac{1}{\ln b} \frac{d}{dx}[\ln x] = \frac{1}{\ln b} \cdot \frac{1}{x} \quad \square$$

ex. DIFFERENTIATE $f(x) = \log_7 \sqrt{x}$

$$f'(x) = \frac{1}{\sqrt{x} \ln 7} \cdot \frac{d}{dx}[\sqrt{x}]$$

$$f'(x) = \frac{1}{\sqrt{x} \ln 7} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2x \ln 7}$$

ex. EVALUATE $\displaystyle\int \frac{\log_{10} x}{x}\, dx$

$$\int (\log_{10} x) \frac{1}{x}\, dx$$

LET $u = \log_{10} x$

$$du = \frac{1}{x \ln 10}\, dx$$

$$\ln 10\, du = \frac{1}{x}\, dx$$

$$\rightarrow \ln 10 \int u\, du = \frac{\ln 10}{2} u^2 + C$$

$$\rightarrow \boxed{\frac{\ln 10}{2} (\log_{10} x)^2 + C}$$

THE NUMBER e AS A LIMIT

SET $f(x) = \ln x$.

THEN $f'(x) = \frac{1}{x}$ & $f'(1) = 1$.

THAT IS, $1 = f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$