Recitation Calculus I (Adamski)

Tuesday, November 17, 2020

Integration by Substitution (u-substitution)

- indefinite case
- definite case

$x=a$
$u=g(a)$

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du, \quad \text{where } \underline{u=g(x)}$$

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Let $\underline{u=g(x)}$

then $\underline{du = g'(x)\,dx}$; $\boxed{\frac{du}{dx} = g'(x)}$

when $x=a$, $u=\underline{\underline{g(a)}}$

$x=b$, $u=\underline{\underline{g(b)}}$

$$\int_{\underline{\underline{a}}}^{\underline{\underline{b}}} \underbrace{f(g(x))}\,\underbrace{g'(x)\,\underline{\underline{dx}}} = \int_{g(a)}^{g(b)} f(u)\,\underline{\underline{du}}$$

$$\int_0^1 x^2\sqrt{1+x^3}\,dx$$

$u = 1+x^3$

$du = 3x^2\,dx$

$\frac{1}{3}du = \underline{x^2\,dx}$

when $x=0$; $u=1$

$x=1$; $u=1+1^3=2$

$$= \int_1^2 \sqrt{u}\,\frac{1}{3}\,\underline{\underline{du}}$$

$$= \frac{1}{3}\int_1^2 u^{1/2}\,du = \frac{1}{3}\left\{\frac{u^{3/2}}{3/2}\right|_1^2 = \frac{2}{9}\left(2^{3/2} - 1^{3/2}\right)$$

$$= \frac{2}{9}\left(2\sqrt{2} - 1\right)$$

$$\frac{1}{3}\cdot\frac{1}{3/2} = \frac{1}{3}\cdot\frac{2}{3} = \frac{2}{9} \checkmark$$

$$\int_1^9 \frac{\sin(\sqrt{x})}{\boxed{\sqrt{x}}}\boxed{dx} \quad 2du$$

Let $u = \sqrt{x} = x^{1/2}$

$du = \frac{1}{2}x^{-1/2}dx = \frac{dx}{2\sqrt{x}}$

$2\,du = \frac{dx}{\sqrt{x}}$

$$= \int_{1\ \cancel{1}}^{3\ \cancel{9}} \sin(u)\,2du$$

when $x = 1$, $u = 1$; $x = 9$, $u = 3$

$$= \int \sin(u)\,2du = 2\left\{-\cos(u)\right|_1^3$$

radians

$$= 2\left\{-\cos(3) - -\cos(1)\right\} = 2\left(\cos(1) - \cos(3)\right)$$

$$\int (\underbrace{1+\sin(x)}) ^{10}\boxed{\cos(x)\,dx} \;;$$

Let $u = 1 + \sin(x)$

$du = \boxed{\cos(x)\,dx}$

$$= \int u^{10}\boxed{du} = \frac{u^{11}}{11} + C$$

$$= \frac{1}{11}\left(1+\sin(x)\right)^{11} + C.$$

Check:

$$\frac{d}{dx}\left(\frac{1}{11}\left(1+\sin(x)\right)^{11} + C\right) = \frac{1}{11}\,11\,(\underbrace{1+\sin(x)})^{10}(0+\cos(x))$$

$$= (1+\sin(x))^{10}\cos(x) \checkmark.$$

Note: Chain Rule is always used check integration by substitution.

Ex

$$\int \frac{\ln(x)}{\boxed{x}}\boxed{dx}$$

Let $u = \ln(x)$

$du = \boxed{\frac{1}{x}dx}$

$$= \int u\,du = \frac{u^2}{2} + C' = \frac{1}{2}(\ln(x))^2 + C'.$$

Check: $\frac{d}{dx} \frac{1}{2}(\ln(x))^2 = \frac{1}{2} 2 (\ln(x))^1 \left(\frac{1}{x}\right)$

$$= \frac{\ln(x)}{x} = \text{integrand} \checkmark.$$

Do you always need to change the boundaries?

<u>Ex</u> $\int_1^{e^5} \frac{\ln(x)}{x}\,dx = \left\{ \frac{1}{2}(\ln(x))^2 \Big|_1^{e^5} \right.$ by FTC

$$= \left\{ \frac{1}{2}(\ln(e^5))^2 - \frac{1}{2}(\ln(1))^2 \right\}$$

indefinite

$$\int \frac{\ln(x)}{\cancel{x}}\,dx = \frac{1}{2}(\ln(x))^2 + C' \qquad = \frac{1}{2}(5^2 - 0) = \frac{25}{2}.$$

Let $u = \ln(x)$ when $x = 1$; $u = 0$

$du = \frac{dx}{x}$ $\qquad x = e^5$; $u = 5$

$\int_1^{e^5} \frac{\ln(x)}{x}\,dx$ definite

$$= \int_0^5 u\,du = \frac{u^2}{2}\Big|_0^5 = \frac{25}{2} \checkmark$$

u = dummy variable

$$= \frac{1}{2}\int_0^1 \frac{\sqrt{u}\,du}{\sqrt{1-u}}$$

<u>Ex</u>

$$\int_0^1 \sqrt{1-x^2}\,dx = \int_1^0 \sqrt{u}\left(-\frac{1}{2}\frac{du}{\sqrt{1-u}}\right)$$

$u = 1 - x^2$; $du = -2x\,dx$; $x^2 = 1 - u$

$w = \sqrt{x}$; $x = w^2$ $\qquad x = \sqrt{1-u}$

$0 \leq x \leq 1$

$$-\frac{1}{2}du = x\,dx = \sqrt{1-u}\,dx$$

$$-\frac{1}{2}\frac{du}{\sqrt{1-u}} = dx$$

$$\int_0^1 \underbrace{\sqrt{1-x^2}}_{\shortparallel}\,dx \quad ; \quad x = w^2$$

when $x=0$; $u = 1-x^2 = 1$

$x = 1$; $u = 1-x^2 = 0$

; $\sqrt{x} = w$

$dx = 2w\,dw =$

$1-x^2 = 1-w^4$

$$\int_0^1 \sqrt{1-w^4}\;2w\,dw$$

when $x=0$; $w=\sqrt{0}=0$

$x=1$; $w=\sqrt{1}=1$

$$= 2\int_0^1 w\sqrt{1-w^4}\,dw$$

<u>Ex</u> $\displaystyle\int_0^1 \sqrt{1-x^2}\,dx$

$u = \sqrt{1-x^2}$; $x = \sqrt{1-u^2}$

$u^2 = 1-x^2$; $x^2 = 1-u^2$

$2u\,du = -2x\,dx$

$u\,du = -x\,dx$

$= -\sqrt{1-u^2}\,dx$

$$-\frac{u\,du}{\sqrt{1-u^2}} = dx$$

$$= \int_1^0 u\left(-\frac{u\,du}{\sqrt{1-u^2}}\right)$$

$$= \int_0^1 \frac{u^2\,du}{\sqrt{1-u^2}}$$

$x=0$; $u=\sqrt{1-0^2}=1$

$x=1$; $u=\sqrt{1-1^2}=0$

<u>Ex</u> $\displaystyle\underbrace{\int_0^1 \sqrt{1-x^2}\,dx}_{\text{area?}} = \frac{\pi}{4}$;

<u>Let</u> $\boxed{x = \sin(u)}$

$\boxed{dx = \cos(u)\,du}$

$x^2 = \sin^2(u)$

$1-x^2 = 1-\sin^2(u) = \cos^2(u)$

$\sqrt{1-x^2} = \cos(u)$ <u>(+)</u>

$$0 \le x \le 1$$

$0 = \sin(0)$, $1 = \sin(\frac{\pi}{2})$

$$= \int_0^{\pi/2} \cos(u)\cos(u)\,du$$

$$= \int_0^{\pi/2} \cos^2(u)\,du$$

$$\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$$

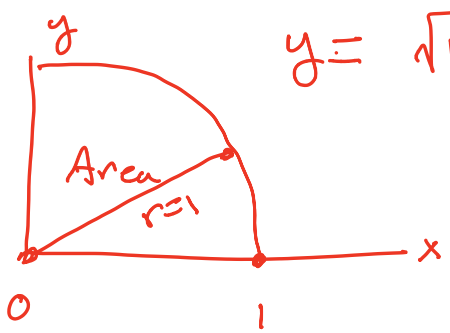
$0 \le u \le \pi/2$

$$= \int_0^{\pi/2} \left(\frac{1}{2} + \frac{1}{2}\cos(2u) \right) du$$

$$= \frac{1}{2}\int_0^{\pi/2} du + \frac{1}{2}\int_0^{\pi/2} \cos(2u)\, du$$

$$= \frac{1}{2}\left\{ u \Big|_0^{\pi/2} + \frac{1}{2}\left\{ \frac{\sin(2u)}{2} \Big|_0^{\pi/2}\right.\right.$$

$$= \frac{1}{2}\left(\frac{\pi}{2}\right) + \frac{1}{4}\left(\sin(\pi) - \sin(0) \right) = \frac{\pi}{4}$$



$y = \sqrt{1-x^2}$; $y^2 = 1 - x^2$;

$x^2 + y^2 = 1$ (✓)

$$\text{Area} = \int_0^1 \sqrt{1-x^2}\, dx = \frac{1}{4}\pi(1)^2$$

$$= \frac{\pi}{4} \checkmark$$

$y = x^3 - 3x + 10$

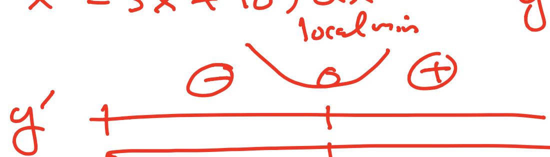
$$\int_0^2 (x^3 - 3x + 10)\, dx$$

$y' = \underline{3x^2 - 3} = 0$

$\Longleftrightarrow$

$x = \pm 1$

$x = 1 \in [0,2]$



$$\boxed{f(1) = 1^3 - 3 + 10 = 8 = m_{\min}}$$

$f(0) = 10$

$f(2) = 8 - 6 + 10 = 12 = M$

$$8(2-0) = m(b-a) \le \int_0^2 (x^3 - 3x + 10)\, dx \le M(b-a) = 12(2-0)$$

$y = -x^2$

$m = y' = -2x = m_1 = -2$

$(a, -a^2)$

$\therefore x = 1$

$x = 1 ; \quad y = -1$