19.

$$x y^2 = \underbrace{\ln(x^4 - 1295)}_{f(x)}$$ TANGENT LINE AT $(6,0)$

EQ OF TANGENT LINE TO $y = f(x)$ AT $(a, f(a))$

DERIVATIVE

$$y - y_1 = m(x - x_1) \rightarrow y - f(a) = f'(a)(x-a)$$

$$y - 0 = f'(6)(x-6)$$

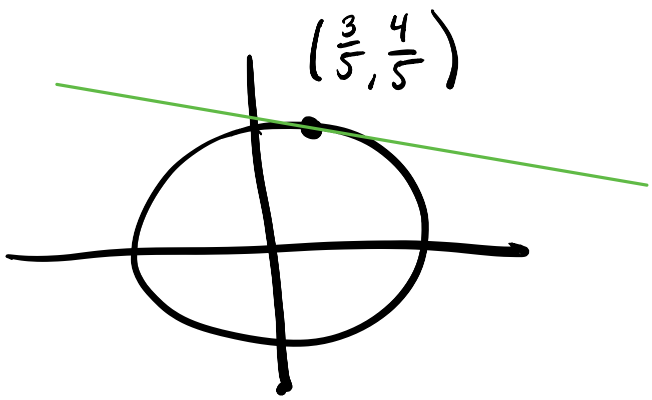
$$f'(x) = \frac{4x^3}{x^4 - 1295} \qquad f'(6) = \ldots$$

$$y - b = \left.\frac{dy}{dx}\right|_{(a,b)} (x-a)$$

IMPLICIT. DIFF.

$$\left.\frac{dy}{dx}\right|_{(a,b)}$$

$$x^2 + y^2 = 1$$



$$y - \frac{4}{5} = \left.\frac{dy}{dx}\right|_{\left(\frac{3}{5}, \frac{4}{5}\right)} \left(x - \frac{3}{5}\right)$$

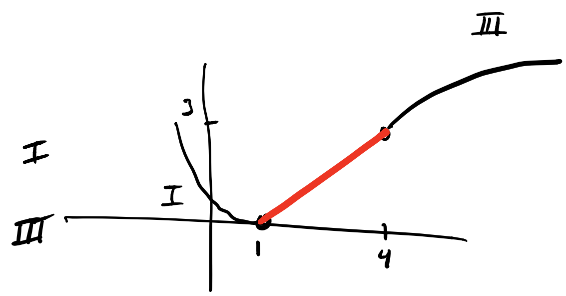
FINAL EXAM MON 12/14 1:30 - 3:30 PM (ZOOM "PAPER" LIKE MIDTERM.)

Fordham Math 1206, Calculus I **Final Exam Practice Problems**

1. Find the limit or state that it does not exist. Justify your answer.
   (a) $\lim_{x\to\infty}\left(\sqrt{x^2+2x}-\sqrt{x^2-x}\right)$
   (b) $\lim_{x\to 0^+}\left(\ln(1/x)-\ln(1/\sin 4x)\right)$
   (c) $\lim_{x\to\infty}\dfrac{\sin x}{\ln x}$

2. Find a and b such that the following function is continuous on $\mathbb{R}$.

$$f(x) = \begin{cases} (x-1)^2 & \text{if } x < 1 \\ ax + b & \text{if } 1 \le x \le 4 \\ \sqrt{2x+1} & \text{if } x > 4 \end{cases}$$



$$\lim_{x \to 1^-} \bigcirc = \lim_{x \to 1^+} \bigcirc = f(1)$$

$$\lim_{x \to 4^-} \bigcirc = \lim_{x \to 4^+} \bigcirc = f(4)$$

3. (a) State the Intermediate Value Theorem, including the hypotheses.

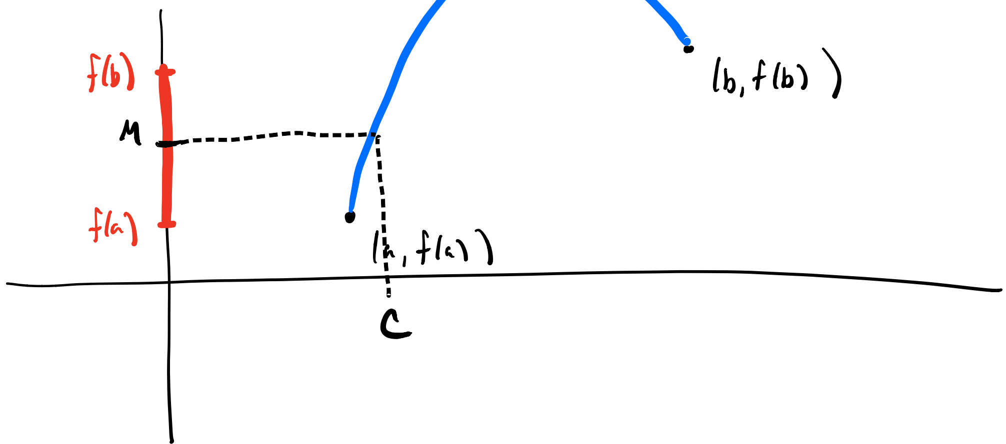
(b) Prove that the following equation has at least one real solution.

$$\sqrt{x-5} = \frac{1}{x+3} \longrightarrow$$

$\dots\dots < 0$

$\dots\dots = 0$

$\dots\dots > 0$



IF f IS CONT. ON $[a,b]$

AND IF $f(a) < M < f(b)$

THEN THERE EXISTS $a < c < b$ SUCH THAT

$f(c) = M$.

$f'(2)$

4. Let $f(x) = \dfrac{2}{3x}$. Use the definition of the derivative as a limit to find $f'(x)$.

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

5. Find $\dfrac{dy}{dx}$.

(a) $y = \sin(x^2)\cos^4(\sqrt{x})$

(b) $y = \dfrac{(2x^5 - 1)^4(3x^4 + 5)^3}{\sqrt{x^2+1}}$

(c) $x^y = y^x$

(d) $y = \int_0^{\tan x} \underbrace{\sqrt{1-t^3}}_{F'(t)}\, dt$ $\rightarrow$ $\dfrac{d}{dx}\left[F(\tan x) - F(0)\right]$

PRODUCT RULE

QUOTIENT RULE

LOGARITHMIC DIFF.

$\sqrt{1-\tan^3(x)}\ \sec^2 x$

(b) $\ln(y) = \ln\left(\dfrac{(2x^5 - 1)^4(3x^4 + 5)^3}{\sqrt{x^2+1}}\right)$

$$\underbrace{\ln(y) = 4\ln(2x^5-1) + 3\ln(3x^4+5) - \tfrac{1}{2}\ln(x^2+1)}_{\text{IMP. DIFF.}}$$

LOG RULES

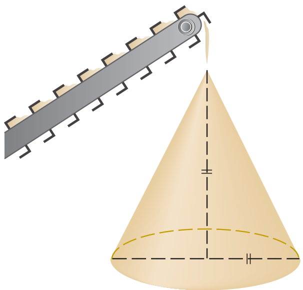
$\dfrac{1}{y}y' = \ldots\ldots \rightarrow y' = y(\ldots\ldots)$

$y' = \dfrac{(2x^5 - 1)^4(3x^4 + 5)^3}{\sqrt{x^2+1}}(\ldots\ldots)$

6. Find the tangent line to the curve $y = \sin(\pi x)e^{-}x$ at the point $(2, 0)$.

RELATED RATES

7. Gravel is being dumped from a conveyer belt at a rate 30 cubic feet per minute, and its courseness is such that it forms a pile in the shape of a cone whose base diameter and height are equal. How fast is the height of the pile increasing when the pile is 10 feet tall?



$$V = \frac{1}{3}\pi r^2 h$$

$$2r = h$$

$$V = \frac{1}{3}\pi r^2 (2r) = \frac{2}{3}\pi r^3$$

$$\frac{d}{dt}[V] = \frac{d}{dt}\left[\frac{2}{3}\pi r^3\right]$$

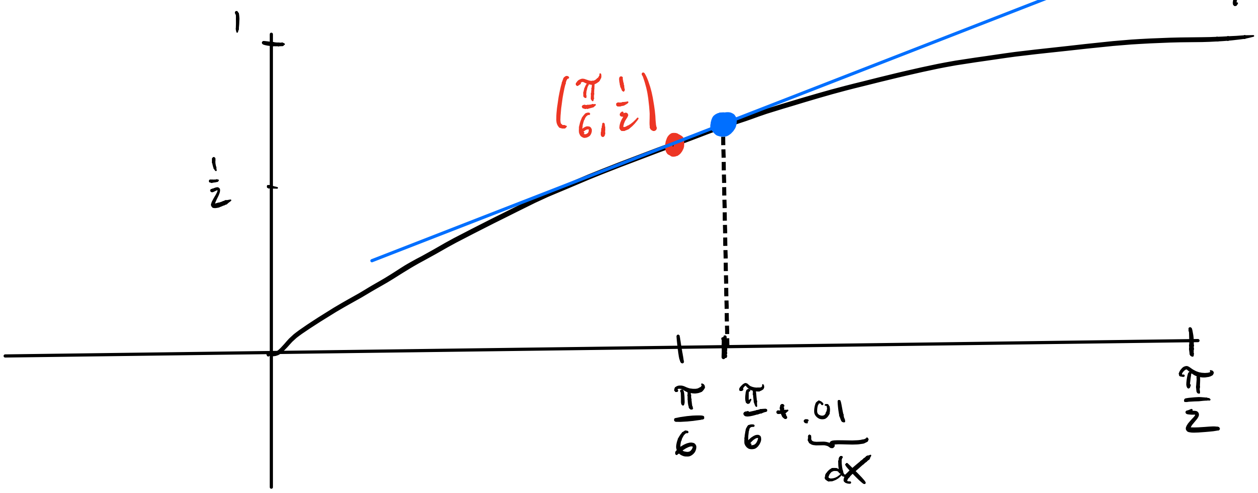
$$\frac{dV}{dt} = \frac{d}{dr}\left[\frac{2}{3}\pi r^3\right]\frac{dr}{dt}$$

8. Use differentials/linear approximation to estimate $\sin(\pi/6 + 0.01)$. You do not need to simplify your answer.

$$y = \frac{1}{2} + \underbrace{\cos\left(\frac{\pi}{6}\right)\overbrace{\left(x - \frac{\pi}{6}\right)}^{dx}}$$

$$y = \sin\left(\frac{\pi}{6}\right)$$

$$dy = \cos x \, dx$$



$$y = \sin(x)$$

$$dy = \cos(x)\, dx$$

9. (a) State the Extreme Value Theorem, including the hypotheses.

(b) Find the absolute maximum and absolute minimum value of $f(x) = \dfrac{\sqrt{x}}{1+x^2}$ over the interval $[0, 2]$

IF f CONT ON $[a,b]$

THEN THERE EXIST c_1 & $c_2 \in [a,b]$

S.T $\underbrace{f(c_1)}_{\text{ABS. MIN VAL.}} \leq f(x)$ FOR ALL $x \in [a,b]$

$\underbrace{f(c_2)}_{\text{ABS. MAX VAL.}} \geq f(x)$ FOR ALL $x \in [a,b]$

CRIT PTS.

END POINTS.

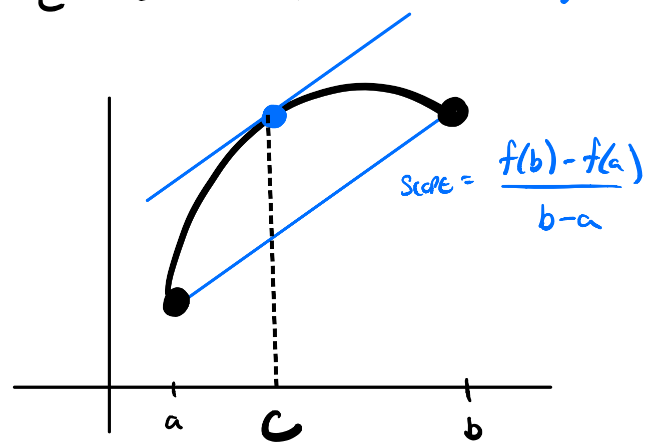
10. (a) State the Mean Value Theorem, including the hypotheses.

(b) Suppose f(x) is a differentiable function such that $f(1) = 2$ and $f'(x) \geq 5$ for all x. What is the smallest possible value for $f(4)$? Justify your answer.

IF f IS CONT ON $[a,b]$ & DIFF. ON (a,b)

THEN THERE IS A POINT $a < c < b$ S.T. SLOPE $f'(c)$

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$



11. Consider the function

$$f(x) = \frac{x^2}{x^2 - 1}.$$

(a) Find the domain of f.

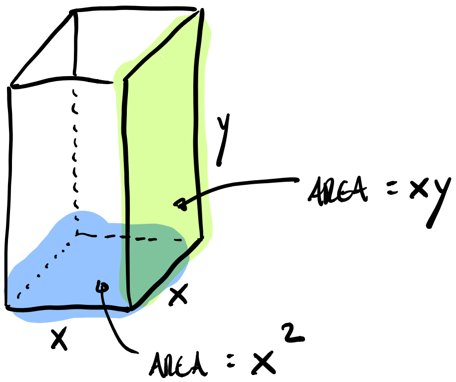
(b) Find the intervals on which f is increasing/decreasing.

(c) Find the intervals on which f is concave up/down.

(d) Sketch the graph $y = f(x)$. Label all intercepts, asymptotes, local maxima/minima, and inflection points.

VERT & HOR.

12. A rectangular box has a square base and an open top. The four sides are made of wood that costs \$2 per square foot, while the base is made of aluminum that costs \$25 per square foot. If the volume of the box is to be 50 cubic feet, what is its minimum possible cost?



$$V = x^2 y \rightarrow 50 = x^2 y \rightarrow x = \sqrt{\frac{50}{y}}$$

OPTIMIZE : MINIMIZE

C = COST OF WOOD + COST OF ALUMINUM

C = (AREA OF SIDES)(COST PER UNIT AREA OF WOOD) + (AREA OF BASE)(COST PER UNIT AREA OF ALUMINUM)

CONSTRAINT:

$$y = \frac{50}{x^2} \qquad x \geq 0 \quad y \geq 0$$

MINIMIZE : $\underbrace{C = (4xy)(2) + (x^2)(25)}_{\text{COST} \,=\, f(x)}$

$$C(x) = 4x \cdot \frac{50}{x^2} \cdot 2 + 25x^2 = \frac{400}{x} + 25x^2$$

$C(x) = \frac{400}{x} + 25x^2$ ← MINIMIZE OVER $(0, \infty)$

IF THERE IS A MIN, IT MUST OCCUR AT CRIT. PNT.

$C'(x) = 0$

$$-\frac{400}{x^2} + 50x = 0$$

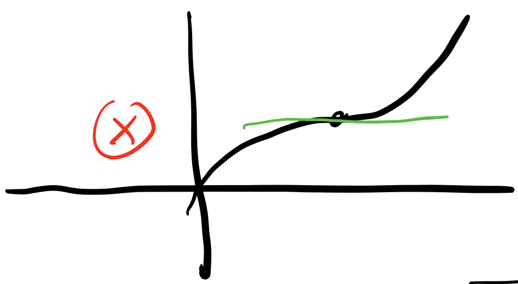
$$-400 + 50x^3 = 0$$

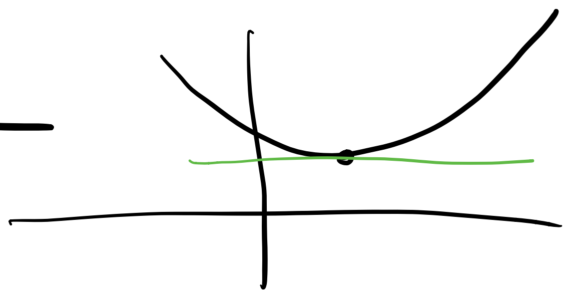
$$x^3 = 8$$

$$x = 2 \longrightarrow y = \frac{50}{x^2} \to \frac{50}{2^2} = \frac{25}{2}$$

$$C(x) = \frac{400}{x} + 25x^2 \longrightarrow C(2) = \frac{400}{2} + 25(2)^2$$

$$= 200 + 100 = \boxed{\$300}$$

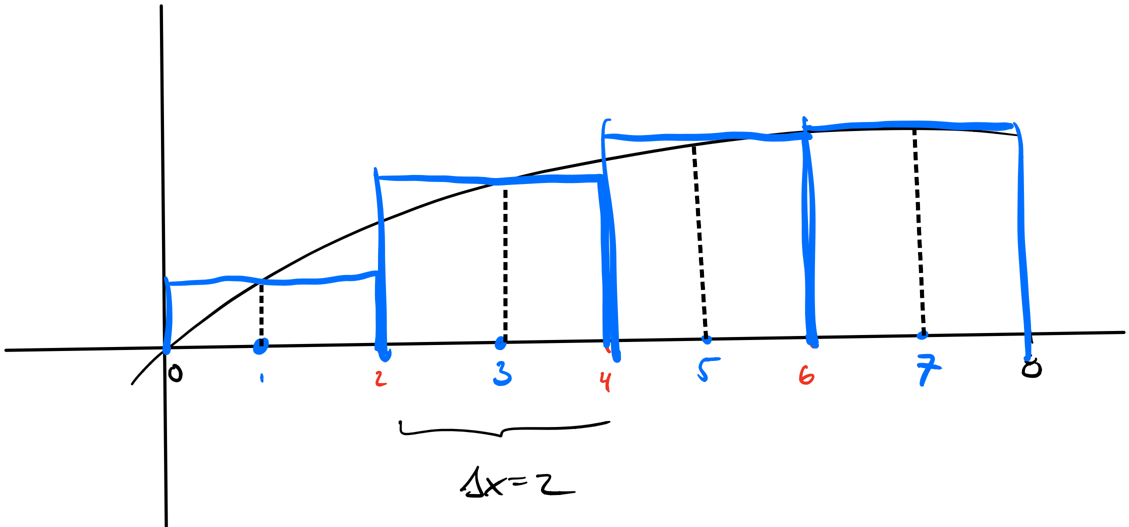




$$C'(x) = -\frac{400}{x^2} + 50x = \frac{50}{x^2}\left(-8 + x^3\right)$$

$$C''(x) = \frac{800}{x^3} + 50 \qquad C''(2) > 0 \quad ✓$$

13. Write down a Riemann sum that estimates $\int_0^8 \sin\sqrt{x}\,dx$ using the midpoint rule with $n = 4$ subdivisions. Do not simplify your answer.



$$\sum_{i=1}^{4} f(\bar{x}_i)\,\Delta x \qquad \Delta x = \frac{b-a}{n} = \frac{8-0}{4} = 2$$

$$= \big(f(1) + f(3) + f(5) + f(7) \big)\,2$$

$$= \big(\sin\sqrt{1} + \sin\sqrt{3} + \sin\sqrt{5} + \sin\sqrt{7} \big)\,2$$

14. Evaluate the integral.

(a) $\displaystyle\int_1^e \frac{\ln(x^6)}{x}\,dx$

(b) $\displaystyle\int \sin^3(3x)\cos(3x)\,dx$

(c) $\displaystyle\int \frac{36}{(2x+1)^3}\,dx$

(d) $\displaystyle\int x^2 2^{x^3}\,dx$

u-SUB

$$\int 2^u\,du = \frac{2^u}{\ln 2} + C$$

$$\frac{d}{du}\left[2^u\right] = 2^u \ln 2$$