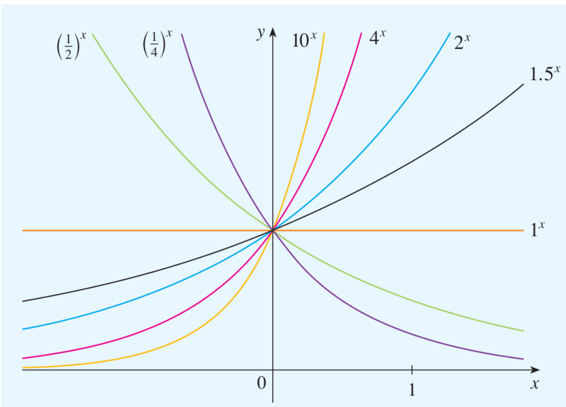
FINAL EXAM MONDAY 12/14 1:30 - 3:30 PM

§6.4* General Logarithmic & Exponential Functions

Given any $b > 0$, we define **the exponential function with base b** to be b^x.



Note: $e^{\ln(b)} = b$

$$\left[e^{\ln(b)}\right]^x = b^x$$

$$\boxed{b^x = e^{(\ln b)x}}$$

If $f(x) = e^x$ then $b^x = e^{(\ln b)x} = f((\ln b)x)$

So the graph $y = b^x$ is the graph $y = e^x$

scaled horizontally

compressed if $\ln(b) > 1 \iff b > e$

stretched if $0 < \ln(b) < 1 \iff 1 < b < e$

If $0 < b < 1$ then $b^{-1} = \frac{1}{b} > 1$

$$b^x = (b^{-1})^{-x} = \left(\frac{1}{b}\right)^{-x}$$

$\frac{1}{b} > 1$

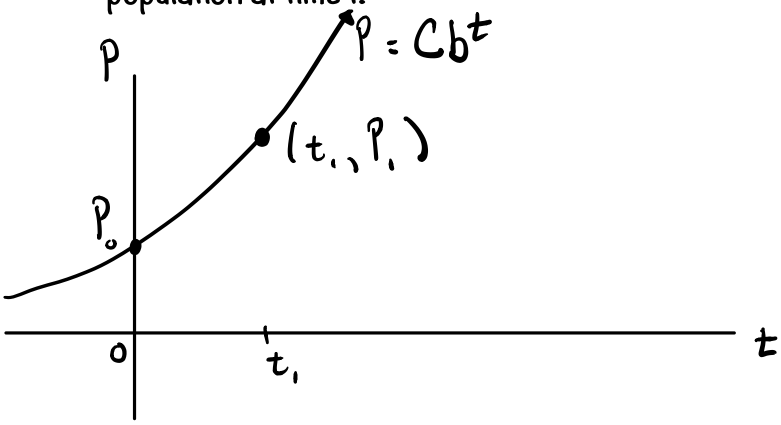
$y = \left(\frac{1}{b}\right)^x$ reflected across y-axis.

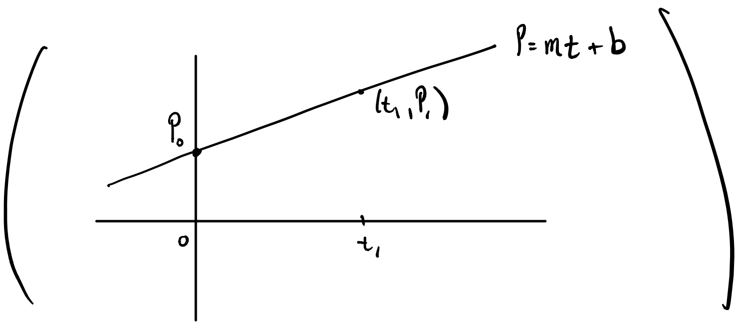
All graphs pass through $(0,1)$: $b^0 = 1$.

ex. Suppose a population grows exponentially. That is, the population at time t is given by

$$P(t) = Cb^t$$

If a population has size P_0 At time t=0 and has size P_1 at time $t = t_1$, find an expression for the population at time t.





$$P(t) = Cb^t$$

C = ? ✓ b = ? ✓

$$P(0) = C\underbrace{b^0}_{1} = P_0$$

$$\underline{\underline{C = P_0}}$$

$$P(t) = P_0 b^t$$

$$P(t_1) = P_0 b^{t_1} = P_1$$

SOLVE FOR b

$$\left(b^{t_1}\right)^{1/t_1} = \left(\frac{P_1}{P_0}\right)^{1/t_1}$$

$$\underline{\underline{b = \left(\frac{P_1}{P_0}\right)^{1/t_1}}}$$

$$P(t) = P_0\left[\left(\frac{P_1}{P_0}\right)^{1/t_1}\right]^t$$

$$\boxed{P(t) = P_0\left(\frac{P_1}{P_0}\right)^{t/t_1}}$$

3 **Laws of Exponents** If x and y are real numbers and $a, b > 0$, then

1. $b^{x+y} = b^x b^y$ **2.** $b^{x-y} = \dfrac{b^x}{b^y}$ **3.** $(b^x)^y = b^{xy}$ **4.** $(ab)^x = a^x b^x$

Proof of 1: $b^{x+y} = \left(e^{\ln b}\right)^{x+y} = e^{\ln(b)x + \ln(b)y}$

$= e^{\ln(b)x} e^{\ln(b)y} = \left[e^{\ln(b)}\right]^x \left[e^{\ln(b)}\right]^y$

$= b^x b^y$

$\square$

$$\frac{d}{dx}(\underbrace{b^x}_{f}) = \underbrace{b^x}_{f'} \overbrace{\ln b}^{c}$$

$f' = cf$ Special case: $c = 1$

$b = e$

Proof: $\frac{d}{dx}(b^x) = \frac{d}{dx}\left(e^{\ln(b)x}\right)$

$= e^{\ln(b)x} \cdot \frac{d}{dx}\left[\ln(b)x\right]$

$= e^{\ln(b)x} \cdot \ln(b) = b^x \cdot \ln b$

$\square$

ex. Differentiate $f(x) = 3^{\cos(2x)}$

$y = e^{2x}$ $y' = e^{2x} \cdot 2$

$f'(x) = 3^{\cos(2x)} \cdot \ln(3) \cdot \frac{d}{dx}\left[\cos(2x)\right]$

$= -3^{\cos(2x)} \cdot \ln(3) \sin(2x)(2) = \boxed{-2\ln(3)\sin(2x)3^{\cos(2x)}}$

ex. Differentiate $f(x) = \tan\left(4^{x^2}\right)$

$f'(x) = \sec^2\left(4^{x^2}\right) \cdot \frac{d}{dx}\left[4^{x^2}\right] = \sec^2\left(4^{x^2}\right) \cdot 4^{x^2} \cdot \ln(4) \frac{d}{dx}\left[x^2\right]$

$= 2\ln(4)\, x \sec^2\left(4^{x^2}\right) 4^{x^2}$

*

ex. DIFFERENTIATE $f(x) = [\tan(x)]^{1/x} = \left[e^{\ln(\tan(x))} \right]^{1/x}$

$$f(x) = e^{\frac{\ln(\tan x)}{x}}$$

$$f'(x) = e^{\frac{\ln(\tan x)}{x}} \cdot \frac{d}{dx}\left[\frac{\ln(\tan x)}{x} \right]$$

$$\boxed{f'(x) = e^{\frac{\ln(\tan x)}{x}} \left(\frac{x \frac{\sec^2 x}{\tan x} - \ln(\tan x)}{x^2} \right)}$$

$$\int b^x \, dx = \frac{b^x}{\ln b} + C \qquad b \neq 1$$

CHECK: $\frac{d}{dx}\left[\frac{b^x}{\ln b} \right] = \frac{1}{\ln b} \frac{d}{dx}\left[b^x \right]$

$= \frac{1}{\cancel{\ln b}} \cdot b^x \cdot \cancel{\ln b}$ ✓

ex. $\int 6^{\sin(3x)} \cos(3x) \, dx$

let $u = \sin(3x)$

$du = 3\cos(3x) \, dx$

$\frac{1}{3} du = \cos(3x) dx$

$\rightsquigarrow \frac{1}{3} \int 6^u \, du = \frac{1}{3} \cdot \frac{6^u}{\ln 6} + C$

$$\rightsquigarrow \boxed{\frac{1}{3 \ln 6} 6^{\sin(3x)} + C}$$

The Power Rule If n is any real number and $f(x) = x^n$, then

$$f'(x) = nx^{n-1}$$

Proof: Let $y = x^n$. Find y'.

$$f(x) = x^{\sqrt{2}}$$
$$f'(x) = \sqrt{2}\, x^{\sqrt{2}-1}$$

$$\ln y = \ln(x^n) = n \ln x$$

$$\frac{d}{dx}[\ln y] = \frac{d}{dx}[n \ln x]$$

$$\frac{1}{y} y' = n \frac{1}{x} \Rightarrow y' = n\frac{y}{x} = n\frac{x^n}{x} = nx^{n-1}$$

$\square$

In general there are four cases for exponents and bases:

Constant base, constant exponent — **1.** $\frac{d}{dx}(b^n) = 0$ (b and n are constants)

Variable base, constant exponent — **2.** $\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1}f'(x)$

Constant base, variable exponent — **3.** $\frac{d}{dx}[b^{g(x)}] = b^{g(x)}(\ln b)g'(x)$

Variable base, variable exponent — **4.** To find $(d/dx)[f(x)]^{g(x)}$, logarithmic differentiation can be used, as in the next example.

EXAMPLE 4 Differentiate $y = x^{\sqrt{x}}$.

$$\ln(y) = \ln(x^{\sqrt{x}}) = \sqrt{x}\ln x$$

$$\frac{d}{dx}\Big($$

$$\frac{1}{y}y' = \frac{1}{2\sqrt{x}}\ln x + \sqrt{x}\frac{1}{x}$$

$$y' = y\left(\frac{1}{2\sqrt{x}}\ln x + \sqrt{x}\frac{1}{x}\right)$$

$$\boxed{y' = x^{\sqrt{x}}\left(\frac{1}{2\sqrt{x}}\ln x + \sqrt{x}\frac{1}{x}\right)}$$

FOR ANY $b > 0$, WE DEFINE THE GENERAL LOGARITHMIC FUNCTION WITH BASE b, $\log_b x$, TO BE THE INVERSE FUNCTION OF THE GENERAL EXPONENTIAL FUNCTION WITH BASE b.

THAT IS,

$$\log_b x = y \iff b^y = x$$

"LOG - BASE - b" $\xleftrightarrow[\text{FUNCTIONS}]{\text{INVERSE}}$ "EXPONENTIAL - BASE - b"

NOTE:

(1) $\log_b b = 1 \iff b^1 = b$

(2) $\log_b 1 = 0 \iff b^0 = 1$

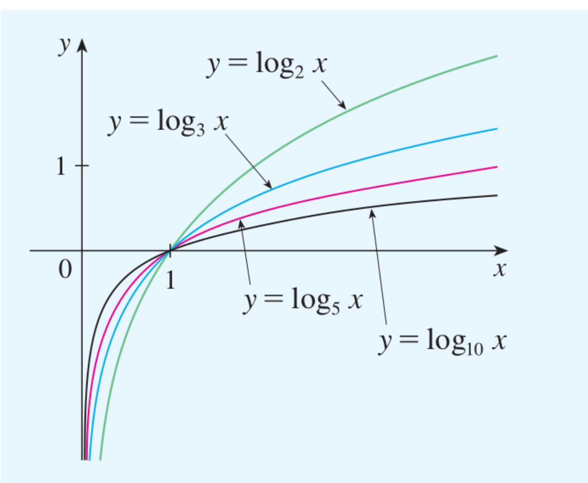
(3) $\log_e x = \ln x$

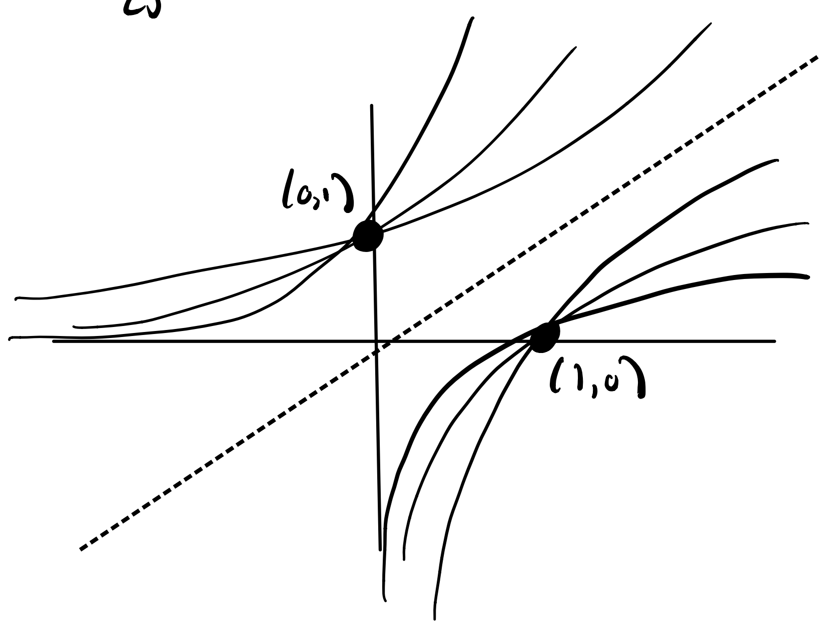
NOTE: $\log_b b^x = x$

$b^{\log_b x} = x$, $x > 0$

ex. $\log_3 27 = x \iff 3^x = 27$ $\quad x = 3$.

ex. $\log_5 \frac{1}{25} = x \iff 5^x = \frac{1}{25}$ $\quad x = -2$





6 **Change of Base Formula** For any positive number b ($b \neq 1$), we have

$$\log_b x = \frac{\ln x}{\ln b}$$

PROOF: Let $y = \log_b x$. $b^y = x$

$$\ln(b^y) = \ln x$$

$$y \ln(b) = \ln x \Rightarrow y = \frac{\ln x}{\ln b} \quad \square$$

$$\left(\log_{2.7} 51 = \frac{\ln(51)}{\ln(2.7)} \right)$$

$$\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$$

ex. DIFFERENTIATE $f(x) = \log_7 \sqrt{x}$

$$f'(x) = \frac{1}{\sqrt{x}\ln 7} \cdot \frac{d}{dx}\left[\sqrt{x}\right] = \frac{1}{\sqrt{x}\ln 7} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2\ln(7)\,x}$$

ex. EVALUATE $\int \frac{\log_{10} x}{x} \, dx$

LET $u = \log_{10} x$

$du = \frac{1}{x \ln 10} \, dx$

$\ln(10) \, du = \frac{1}{x} \, dx$

$$\int \log_{10} x \cdot \frac{1}{x} \, dx \rightsquigarrow \ln(10) \int u \, du = \frac{\ln 10}{2} u^2 + C$$

$$\rightsquigarrow \boxed{\frac{\ln 10}{2} \left(\log_{10} x \right)^2 + C}$$

THE NUMBER e AS A LIMIT

$$\boxed{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e}$$

SET $f(x) = \ln x$.
THEN $f'(x) = \frac{1}{x}$ & $f'(1) = 1$.

THAT IS, $1 = f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$ DEF. OF DERIV.

$$= \lim_{h\to 0} \frac{1}{h}\Big(\ln(1+h) - \underbrace{\ln(1)}_{0}\Big) = \lim_{h\to 0} \frac{1}{h} \ln(1+h)$$

$$\therefore\ 1 = \lim_{h\to 0} \frac{1}{h}\ln(1+h)$$

$$\therefore\ 1 = \lim_{x\to 0} \frac{1}{x}\ln(1+x) = \lim_{x\to 0} \ln\left((1+x)^{1/x}\right)$$

$$e^1 = e^{\lim_{x\to 0} \ln\left((1+x)^{1/x}\right)}$$

e^x IS CONTINUOUS!
LIMITS CAN PASS THROUGH CONTINUOUS FUNCTIONS.

$$e = \lim_{x\to 0} e^{\ln\left((1+x)^{1/x}\right)} = \lim_{x\to 0} (1+x)^{1/x}$$

$$e = \lim_{x\to 0} (1+x)^{1/x} \Rightarrow e = \lim_{x\to 0^+} (1+x)^{1/x}$$

LET $n = \frac{1}{x} \longleftrightarrow x = \frac{1}{n}$.
AS $x \to 0^+$, $n \to \infty$

$$\therefore\ e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

$\square$