# § 4.2 The Definite Integral

**2 Definition of a Definite Integral** If $f$ is a function defined for $a \leqslant x \leqslant b$, we divide the interval $[a, b]$ into $n$ subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0\ (= a), x_1, x_2, \ldots, x_n\ (= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any **sample points** in these subintervals, so $x_i^*$ lies in the $i$th subinterval $[x_{i-1}, x_i]$. Then the **definite integral of $f$ from $a$ to $b$** is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that $f$ is **integrable** on $[a, b]$.

Anatomy:

Integral sign "S" for limit of sums

Upper limit/bound

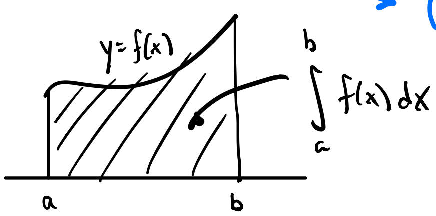
$$\int_a^b f(x)\,dx$$

Differential

For now: bookend that specifies the independent variable

Lower limit/bound

Integrand




$$= \underbrace{\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x}_{\text{Riemann Sum}}$$

**Riemann**

Bernhard Riemann received his Ph.D. under the direction of the legendary Gauss at the University of Göttingen and remained there to teach. Gauss, who was not in the habit of praising other mathematicians, spoke of Riemann's "creative, active, truly mathematical mind and gloriously fertile originality." The definition (2) of an integral that we use is due to Riemann. He also made major contributions to the theory of functions of a complex variable, mathematical physics, number theory, and the foundations of geometry. Riemann's broad concept of space and geometry turned out to be the right setting, 50 years later, for Einstein's general relativity theory. Riemann's health was poor throughout his life, and he died of tuberculosis at the age of 39.

The precise meaning of the limit that defines the integral is as follows:

For every number $\varepsilon > 0$ there is an integer $N$ such that

$$\left| \int_a^b f(x)\,dx - \sum_{i=1}^{n} f(x_i^*)\,\Delta x \right| < \varepsilon$$

for every integer $n > N$ and for every choice of $x_i^*$ in $[x_{i-1}, x_i]$.

Note: $\int_a^b f(x)\,dx$ IS A LIMIT.

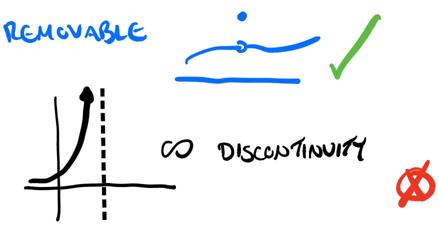
$\underbrace{\text{IF IT EXISTS}}$, IT IS A REAL NUMBER.

Def: $f$ IS INTEGRABLE IF $\int_a^b f(x)\,dx$ EXIST

FOR ALL $a, b \in \mathbb{R}$ SUCH THAT $[a,b] \subseteq \mathrm{Dom}(f)$.

$\downarrow$

OR REMOVABLE



$\infty$ DISCONTINUITY

**3 Theorem** If $f$ is continuous on $[a, b]$, or if $f$ has only a finite number of jump discontinuities, then $f$ is integrable on $[a, b]$; that is, the definite integral $\int_a^b f(x)\,dx$ exists.

If $f$ is integrable on $[a, b]$, then the limit in Definition 2 exists and gives the same value no matter how we choose the sample points $x_i^*$. To simplify the calculation of the integral we often take the sample points to be right endpoints. Then $x_i^* = x_i$ and the definition of an integral simplifies as follows.

https://www.geogebra.org/m/CfwjsmHx

**4 Theorem** If $f$ is integrable on $[a, b]$, then

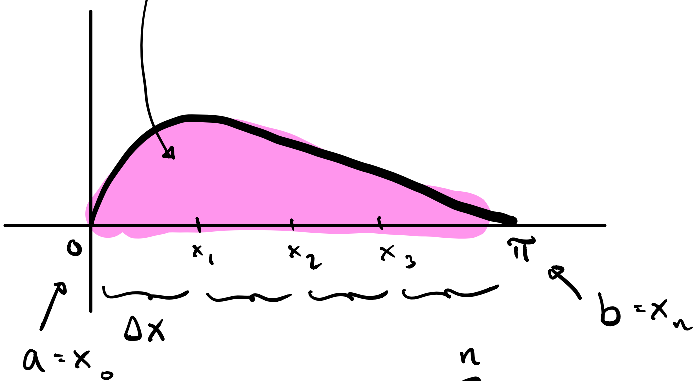
$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

where $\qquad \Delta x = \dfrac{b-a}{n} \qquad$ and $\qquad x_i = a + i\,\Delta x$

ex. WRITE THE DEFINITE INTEGRAL AS A LIMIT OF RIEMANN SUMS:

$$\int_0^{\pi} \frac{\sin x}{1+x^2}\,dx$$

= SIGNED AREA BETWEEN $y = \frac{\sin x}{1+x^2}$, $y=0$, $x=0$, $x=\pi$.




$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

$$f(x) = \frac{\sin x}{1+x^2}$$

$$\Delta x = \frac{b-a}{n} = \frac{\pi - 0}{n} = \frac{\pi}{n}$$

$$x_i = a + i\Delta x = 0 + i\left(\frac{\pi}{n}\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{\sin\left(\frac{\pi i}{n}\right)}{1+\left(\frac{\pi i}{n}\right)^2}\left(\frac{\pi}{n}\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi \sin\left(\frac{\pi i}{n}\right)}{n + \frac{\pi^2 i^2}{n}} = \ldots$$

ex. EXPRESS THE LIMIT AS A DEFINITE INTEGRAL

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3\sqrt{\left(2+\frac{3i}{n}\right)}\ \sec^2\left(2+\frac{3i}{n}\right)}{n}$$

$b - a = b - 2 = 3 \Rightarrow b = 5$

LET $f(x) = \sqrt{x}\ \sec^2 x$

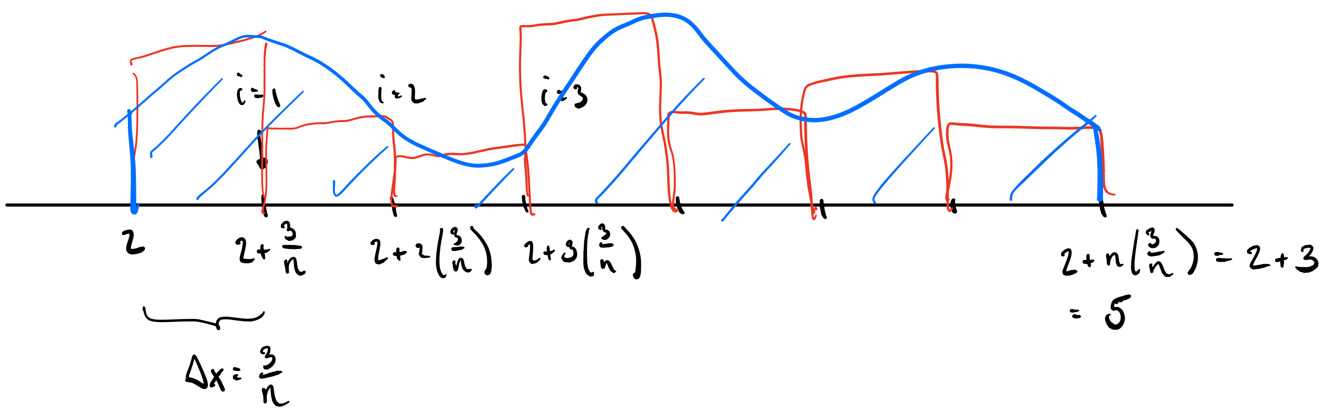
$\Delta x = \frac{b-a}{n} = \frac{3}{n}$

$$= \lim_{n\to\infty} \left[ \sum_{i=1}^{n} f\Big(\underbrace{2+\frac{3i}{n}}_{x_i}\Big) \cdot \underbrace{\frac{3}{n}}_{\Delta x} \right]$$

$x_i = a + i\Delta x = 2 + i\frac{3}{n} = 2 + i\Delta x$

$a = 2$

WHAT IS BEING PLUGGED INTO $f$ ?




$$= \int_2^5 f(x)\,dx = \int_2^5 \sqrt{x}\ \sec^2(x)\,dx$$

**5** $$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

**6** $$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

**7** $$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

The remaining formulas are simple rules for working with sigma notation:

**8** $$\sum_{i=1}^{n} c = nc$$

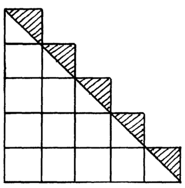
**9** $$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$$

**10** $$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

**11** $$\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$$

**Proof without words:**
**Sum of integers**

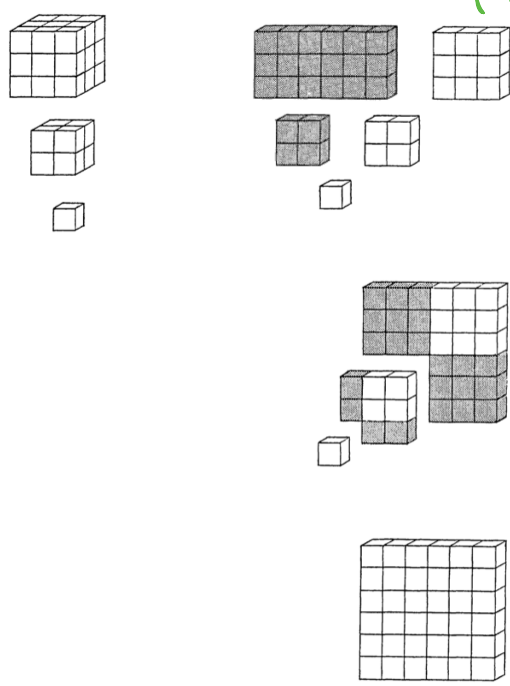
$$1+2+3+\cdots+n=\frac{n^2}{2}+\frac{n}{2} = \frac{n(n+1)}{2}$$



—Ian Richards
University of Minnesota
Minneapolis, MN 55455

**Proof without words:**
**Sum of cubes**

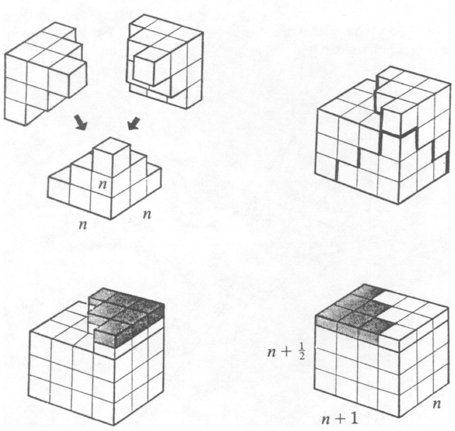
$$1^3+2^3+\cdots+n^3=(1+2+\cdots+n)^2 = \left(\frac{n(n+1)}{2}\right)^2$$



—Alan L. Fry
University of North Carolina
at Greensboro

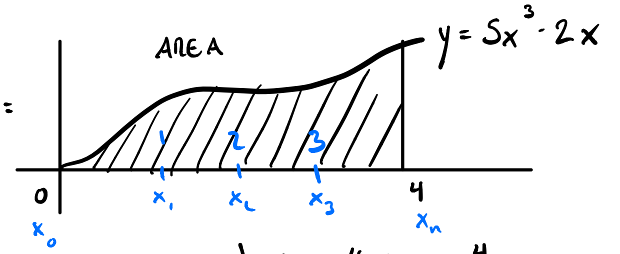
**Proof without words:**
**Sum of squares**

$$1^2+2^2+\cdots+n^2=\tfrac{1}{3}n(n+1)(n+\tfrac{1}{2})$$




—Man-Keung Siu
University of Hong Kong

ex. EVALUATE $\int_0^4 5x^3 - 2x \, dx$ =




$= \int_a^b f(x)\,dx$

$f(x) = 5x^3 - 2x$

$a = 0 \qquad b = 4$

$\Delta x = \frac{b-a}{n} = \frac{4-0}{n} = \frac{4}{n}$

$x_i = a + i\Delta x = 0 + i\frac{4}{n} = \frac{4i}{n}$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{\left(5(x_i)^3 - 2(x_i)\right)}_{f(x_i)} \underbrace{\frac{4}{n}}_{\Delta x}$$

$$= \lim_{n\to\infty} \left[ \sum_{i=1}^{n} \left(5\left(\frac{4i}{n}\right)^3 - 2\left(\frac{4i}{n}\right)\right)\frac{4}{n} \right]$$

n IS BEING TREATED LIKE A CONSTANT INSID [ ]

$$= \lim_{n\to\infty} \left[ \frac{4}{n} \sum_{i=1}^{n} \left(5 \cdot \frac{4^3}{n^3} i^3 - 2 \cdot \frac{4}{n} i\right) \right]$$

FORMULAS!

$$= \lim_{n\to\infty} \left[ \frac{4}{n} \cdot 5 \cdot \frac{4^3}{n^3} \sum_{i=1}^{n} i^3 - \frac{4}{n} \cdot 2 \cdot \frac{4}{n} \sum_{i=1}^{n} i \right]$$

$$= \lim_{n\to\infty} \left[ \frac{5 \cdot 4^4}{n^4} \left(\frac{n(n+1)}{2}\right)^2 - \frac{2 \cdot 4^2}{n^2} \cdot \frac{n(n+1)}{2} \right]$$

i's HAVE GONE AWAY

$$= \lim_{n\to\infty} \frac{5\cdot 4^4}{4} \cdot \frac{n^4+2n^3+n^2}{n^4} - \lim_{n\to\infty} \frac{2\cdot 4^2}{2} \frac{n^2+n}{n^2}$$

$$= \frac{5\cdot 4^4}{4} \lim_{n\to\infty} \frac{n^4+2n^3+n^2}{n^4} - \frac{2\cdot 4^2}{2} \lim_{n\to\infty} \frac{n^2+n}{n^2}$$

$$= 5\cdot 4^3 \lim_{n\to\infty} \left( \frac{n^4}{n^4} + \cancelto{0}{\frac{2n^3}{n^4}} + \cancelto{0}{\frac{n^2}{n^4}} \right) - 4^2 \lim_{n\to\infty} \left( 1 + \cancelto{0}{\frac{1}{n}} \right)$$

$$= 5\cdot 4^3(1) - 4^2(1) = 320 - 16 = 304$$

ex. EVALUATE $\int_{-3}^{3} \sqrt{9 - x^2}\, dx$

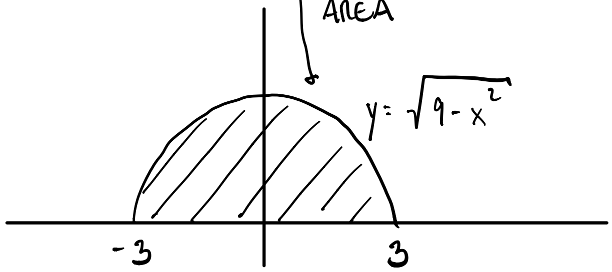
INTEGRAND: $f(x) = \sqrt{9 - x^2}$ , $-3 \le x \le 3$

GRAPH: $y = \sqrt{9 - x^2}$ , $y \ge 0$

$y^2 = 9 - x^2$

$x^2 + y^2 = 9$ , $y \ge 0$

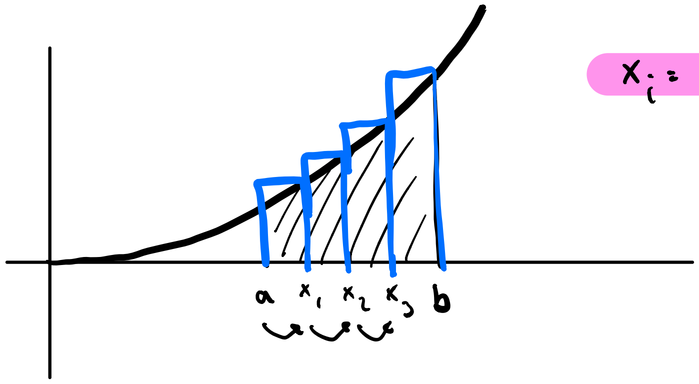
AREA




$$\text{AREA} = \frac{1}{2}\left( \pi \cdot 3^2 \right) = \frac{9\pi}{2}$$

ex. Prove that $\int_a^b x^2\,dx = \dfrac{b^3-a^3}{3}$

$f(x) = x^2$ $\quad \Delta x = \dfrac{b-a}{n}$




$x_i = a + i\Delta x = a + \dfrac{i(b-a)}{n}$

$$\int_a^b x^2\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} (x_i)^2 \Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(a + \frac{(b-a)}{n} i\right)^2 \left(\frac{b-a}{n}\right)$$

$$= \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} \left(a^2 + \frac{2a(b-a)}{n} i + \left(\frac{b-a}{n}\right)^2 i^2\right)$$

$$= \lim_{n\to\infty} \left[ \frac{a^2(b-a)}{n} \sum_{i=1}^{n} 1 + \frac{2a(b-a)^2}{n^2} \sum_{i=1}^{n} i + \left(\frac{b-a}{n}\right)^3 \sum_{i=1}^{n} i^2 \right]$$

$$= \lim_{n\to\infty} \left[ \frac{a^2(b-a)}{n} n + \frac{2a(b-a)^2}{n^2} \cdot \frac{n(n+1)}{2} + \frac{(b-a)^3}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} \right]$$

$$= \lim_{n\to\infty} \left[ a^2(b-a) + a(b-a)^2\left(1 + \frac{1}{n}\right) + \frac{(b-a)^3}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) \right]$$

$$= a^2(b-a) + a(b-a)^2 + \frac{(b-a)^3}{3}$$

$$= \frac{3a^2(b-a) + 3a(b-a)^2 + (b-a)(b^2-2ab+a^2)}{3}$$

$$= \frac{(b-a)\left[3a^2 + 3a(b-a) + b^2 - 2ab + a^2\right]}{3}$$

$$= \frac{(b-a)\left[3a^2 + 3ab - 3a^2 + b^2 - 2ab + a^2\right]}{3}$$

$$= \frac{(b-a)\left[b^2 + ab + a^2\right]}{3} = \frac{b^3 - a^3}{3}$$