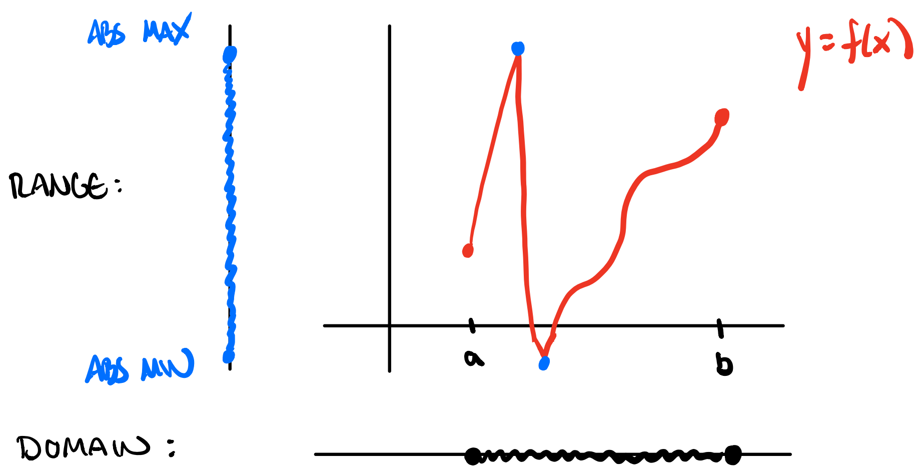
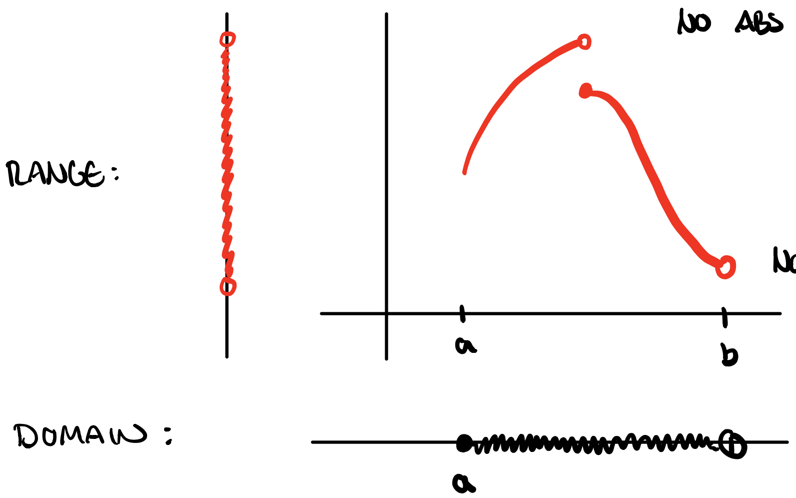
§3.2 THE MEAN VALUE THEOREM

RECALL:

3 The Extreme Value Theorem If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

VERY INTUITIVE, BUT THE CONDITIONS ARE NECESSARY.





NO ABS MAX: f IS NOT CONTINUOUS

NO ABS MIN: DOMAIN IS NOT A CLOSED INTERVAL.

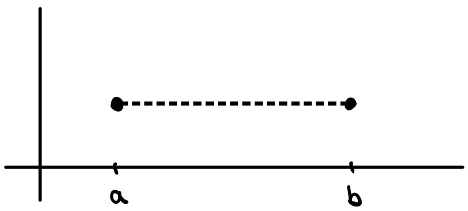
Rolle's Theorem Let f be a function that satisfies the following three hypotheses:

1. f is continuous on the closed interval $[a, b]$. (EXTREME VAL. THM APPLIES)
2. f is differentiable on the open interval (a, b).
3. $f(a) = f(b)$

Then there is a number c in (a, b) such that $f'(c) = 0$.

PROOF: 1. IF $f(x) = k$ CONSTANT $\Rightarrow$ OBVIOUS.

2. ELSE, $f(x) \gtrless f(a) = f(b)$ FOR SOME $x \in (a,b)$. $(*)$



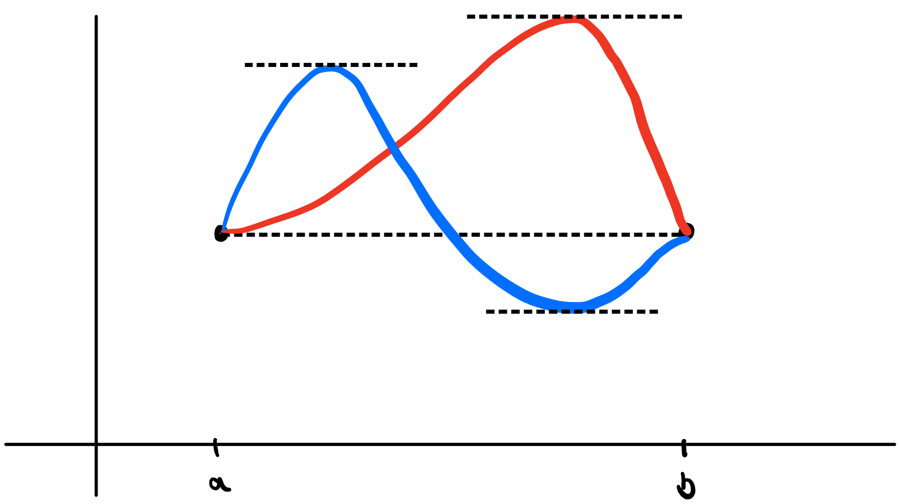
EXTREME VALUE THM $\Rightarrow$ $\exists\, c \in [a,b]$ SUCH THAT

$f(c)$ IS AN ABSOLUTE MAX/MIN.

$(*) \Rightarrow c \in (a,b) \Rightarrow c$ IS LOCAL MAX/MIN.

FERMAT'S THM $\Rightarrow f'(c) = 0$.

$\square$



19–20 Show that the equation has exactly one real root.

19. $2x + \cos x = 0$ **20.** $2x - 1 - \sin x = 0$

EXISTENCE

UNIQUENESS

LET $f(x) = 2x + \cos x$

$f'(x) = 2 - \sin x$; $1 \leq f'(x) \leq 3$

ASSUME WE HAVE 2 #'s a, b SUCH THAT

$f(a) = f(b) = 0$, $a < b$.

1 f IS CONTINUOUS ON $[a, b]$

2. f IS DIFFERENTIABLE ON (a, b)

3. $f(a) = f(b)$

ROLLE'S THM $\Rightarrow$ $\exists c \in (a,b)$ S.T.

$f'(c) = 0$

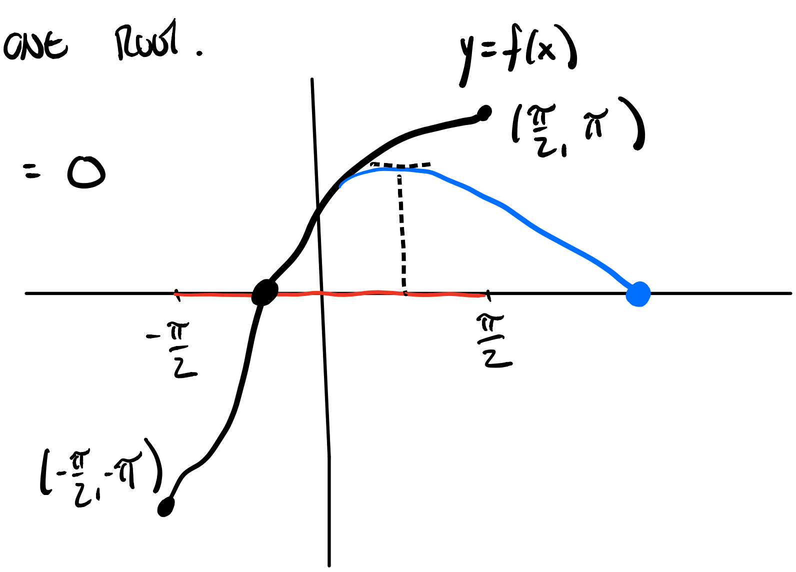
BUT $f'(x) \neq 0 \quad \forall x \in \mathbb{R}$

$\Rightarrow\!\Leftarrow$

THERE CANNOT BE TWO #'s $a \neq b$ S.T. $f(a) = f(b) = 0$.

NOW SHOW THERE IS ONE ROOT.

$$f(x) = 2x - \cos x = 0$$



x	$f(x)$
$-\pi/2$	$-\pi$
$\pi/2$	π

SINCE f IS CONTINUOUS ON $[-\frac{\pi}{2}, \frac{\pi}{2}]$ &

$$-\pi = \underbrace{f(-\tfrac{\pi}{2}) < 0 < f(\tfrac{\pi}{2})}= \pi$$

$\exists\, c \in (-\frac{\pi}{2}, \frac{\pi}{2})$ S.T.

$$f(c) = 0$$

The Mean Value Theorem Let f be a function that satisfies the following hypotheses:

1. f is continuous on the closed interval $[a, b]$.

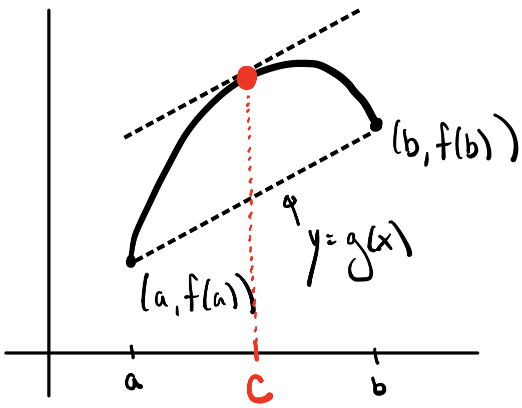
2. f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

$$\boxed{1} \qquad f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently,

$$\boxed{2} \qquad f(b) - f(a) = f'(c)(b - a)$$



SLOPE OF SECANT LINE FROM $(a, f(a))$ TO $(b, f(b))$

$$\frac{f(b) - f(a)}{b - a}$$

← AVERAGE RATE OF CHANGE

EQUALS SLOPE OF TANGENT LINE SOMEWHERE BETWEEN.

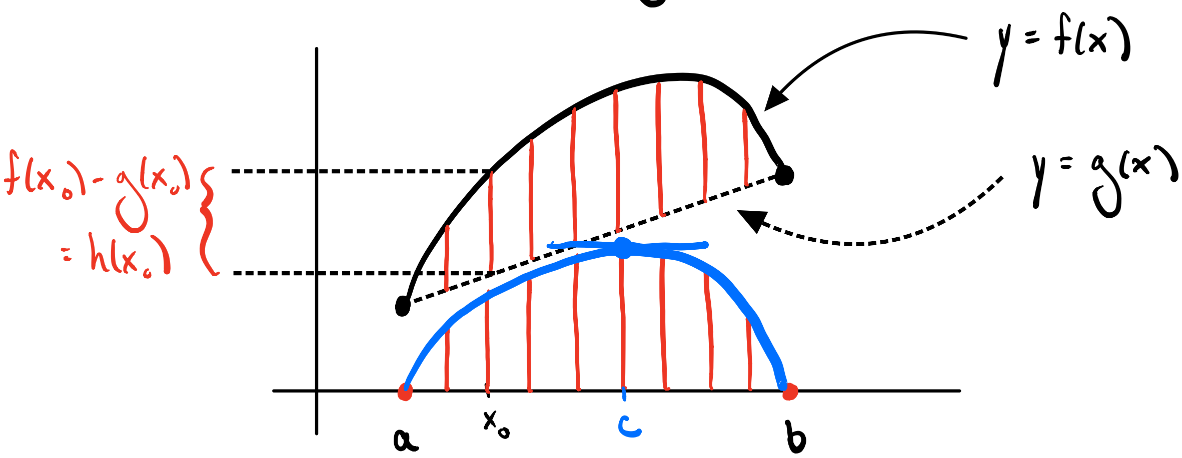
$$f'(c)$$

← INSTANTANEOUS RATE OF CHANGE

PROOF: SET $\underbrace{g(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)}$

GRAPH IS LINE THROUGH $(a, f(a))$ & $(b, f(b))$

SET $h(x) = f(x) - g(x)$



1. h IS CONTINUOUS ON $[a, b]$
2. h IS DIFFERENTIABLE ON (a, b)
3. $h(a) = h(b) = 0$

ROLLES' THM $\Rightarrow \exists\ c \in (a, b)$ SUCH THAT $h'(c) = 0$.

SINCE $h'(c) = f'(c) - g'(c)$ WE HAVE

$$f'(c) = g'(c) = \frac{f(b) - f(a)}{b - a}. \qquad \square$$

ex. SUPPOSE $f(0) = 5$ & $f'(x) = 4$ $\forall x \in \mathbb{R}$.

WHAT CAN YOU SAY ABOUT $f(x)$?

LET $x \in \mathbb{R}$, $x > 0$.

(1) f IS CONTINUOUS ON $[0, x]$

(2) f IS DIFFERENTIABLE ON $(0, x)$

$\therefore$ MVT $\Rightarrow$ $\exists c \in (0, x)$ S.T.

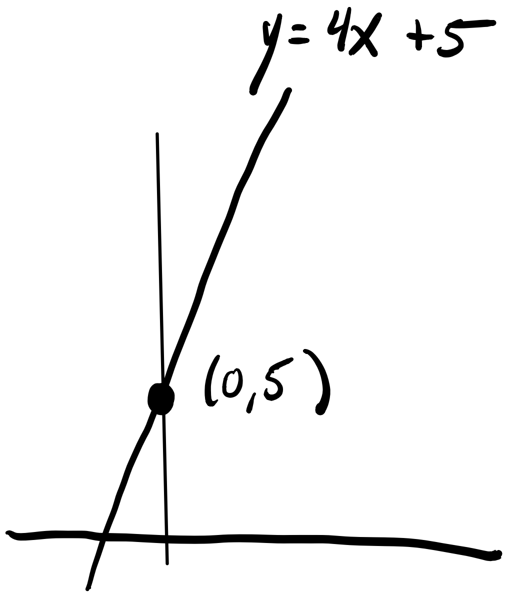
$$\frac{f(x) - f(0)}{x - 0} = f'(c)$$ BUT $f'(x) = 4$ $\forall x$!

$$\frac{f(x) - f(0)}{x - 0} = 4$$

$$f(x) - f(0) = 4(x - 0)$$

$$f(x) - 5 = 4x$$

$$\boxed{f(x) = 4x + 5}$$



5 Theorem If $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).

PROOF: $\forall\, x_1, x_2$ SATISFYING $a < x_1 < x_2 < b$ SUBSET

1. f IS CONTINUOUS ON $[x_1, x_2] \subseteq [a,b]$
2. f IS DIFFERENTIABLE ON $(x_1, x_2) \subseteq (a,b)$

MVT $\Rightarrow \exists\, c \in (x_1, x_2)$ SUCH THAT

$$f(x_2) - f(x_1) = \underbrace{f'(c)}_{=\,0}(x_2 - x_1)$$

$\therefore f(x_1) = f(x_2) \;\; \forall\, x_1, x_2 \in (a,b)$,

i.e. f IS CONSTANT ON (a,b). $\square$

(NOTE: IF WE ASSUME f IS CONTINUOUS ON $[a,b]$
AND $f'(x) = 0 \;\; \forall\, x \in (a,b)$
THEN $f(x)$ IS CONSTANT ON $[a,b]$.)

FOLLOWS IMMEDIATELY FROM PREV. THM.

$\downarrow$

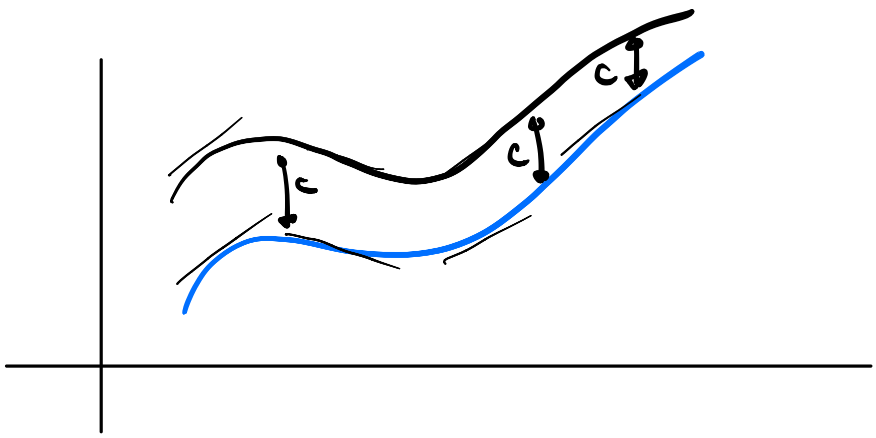
7 Corollary If $f'(x) = g'(x)$ for all x in an interval (a, b), then $f - g$ is constant on (a, b); that is, $f(x) = g(x) + c$ where c is a constant.

PROOF: SET $h(x) = f(x) - g(x)$

$h'(x) = f'(x) - g'(x) = 0$ BY ASSUMPTION.

$\Rightarrow$ $h(x) = c$ CONSTANT

$\therefore$ $f(x) - g(x) = c \Rightarrow f(x) = g(x) + c$ $\square$



11–14 Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

- **11.** $f(x) = 2x^2 - 3x + 1$, $[0, 2]$

12. $f(x) = x^3 - 3x + 2$, $[-2, 2]$

13. $f(x) = \sqrt[3]{x}$, $[0, 1]$

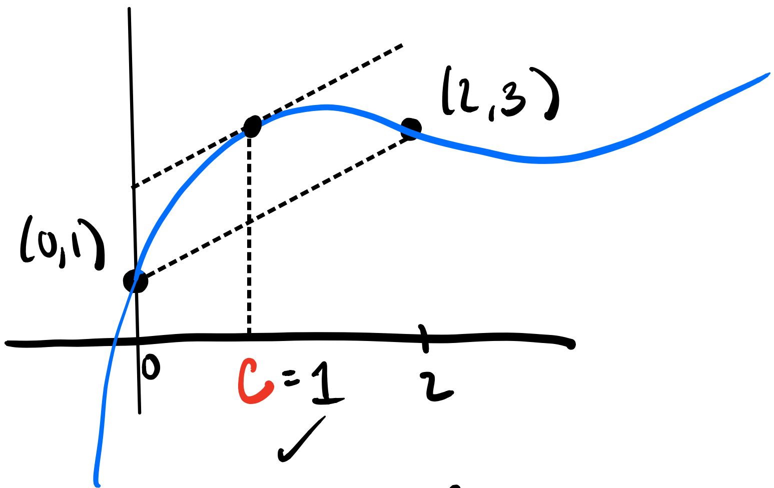
14. $f(x) = 1/x$, $[1, 3]$

11. f IS CONTINUOUS ON $[0,2]$ (POLYNOMIALS CONT. EVERYWHERE)

f IS DIFF. ON $(0,2)$ (POLYNOMIALS DIFF EVERYWHERE)

MVT $\Rightarrow$ $\exists\, c \in (0,2)$ S.T.

$$f'(c) = \frac{f(2) - f(0)}{2 - 0} = \frac{3 - 1}{2 - 0} = 1$$



$$f(x) = 2x^2 - 3x + 1$$

$$f'(x) = 4x - 3 \qquad = 1$$

$$4x = 4$$

$$x = 1 \quad ✓$$

$$\downarrow$$

$$0 < 1 < 2$$

§3.3 How Derivatives Affect the Shape of a Graph

Increasing/Decreasing Test

(a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.

(b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

PROOF: TAKE $a, b \in I$ WITH $a < b$. $(a,b) \subseteq [a,b] \subseteq I$

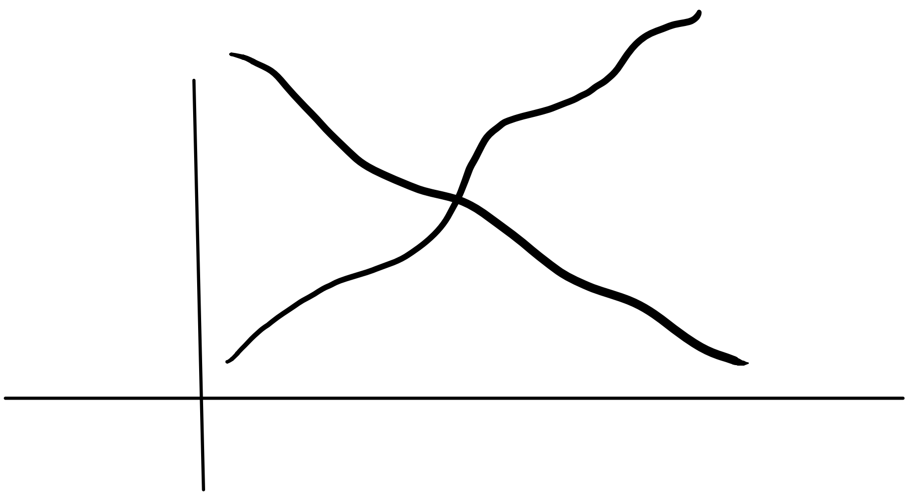
MVT. $\Rightarrow$ $f(b) - f(a) = f'(c)\underbrace{(b-a)}_{\text{POSITIVE}}$ FOR SOME $c \in (a,b)$.

IF $f'(c) > 0$ THEN $f(b) - f(a) > 0$, i.e. $f(b) > f(a)$.

IF $f'(c) < 0$ THEN $f(b) - f(a) < 0$, i.e. $f(b) < f(a)$.

SINCE THIS HOLDS FOR ALL $a < b$ IN THE INTERVAL I,

f IS INCREASING/DECREASING ON I. $\square$



ex. FIND THE INTERVAL(S) ON WHICH

$$f(x) = x^4 - 4x^3 + 4x^2$$

IS INCREASING/DECREASING.

FIND THE INTERVAL(S) ON WHICH $f'(x)$ IS POS/NEG.

$$f'(x) = 4x^3 - 12x^2 + 8x = 4x(x^2 - 3x + 2)$$

$$= 4x(x-2)(x-1) = 0$$

$f'(x) = 0$ AT $x = 0$, $x = 1$, $x = 2$

	$(-\infty, 0)$	$(0, 1)$	$(1, 2)$	$(2, \infty)$
f'	0	1	2	

$4x$	NEG	POS	POS	POS
$x-2$	NEG	NEG	NEG	POS
$x-1$	NEG	NEG	POS	POS
$4x(x-2)(x-1)$	NEG	POS.	NEG	POS

$f'(x)$ IS NEGATIVE ON $(-\infty, 0) \cup (1, 2)$

$f'(x)$ IS POSITIVE ON $(0, 1) \cup (2, \infty)$

$f(x)$ IS DECREASING ON $(-\infty, 0) \cup (1, 2)$

$f(x)$ IS INCREASING ON $(0, 1) \cup (2, \infty)$