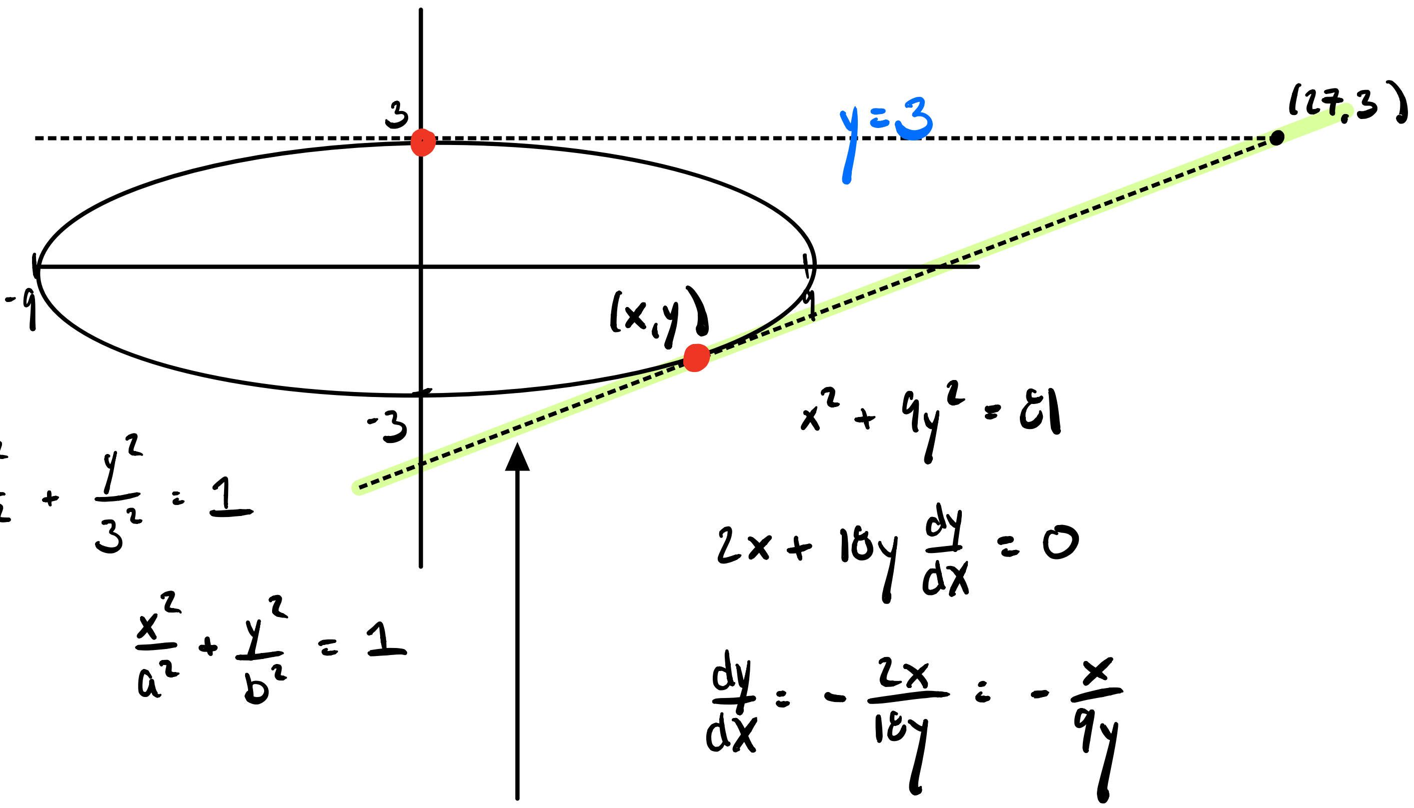
Find equations of both the tangent lines to the ellipse $x^2 + 9y^2 = 81$ that pass through the point $(27, 3)$.

$y =$ 3 $\boxed{3}$ (smaller slope)

$y =$ $\boxed{\frac{1}{4}x - \frac{15}{4}}$ (larger slope)



$$\frac{x^2}{9^2} + \frac{y^2}{3^2} = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$x^2 + 9y^2 = 81$$

$$2x + 18y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{2x}{18y} = -\frac{x}{9y}$$

PASSES THROUGH $(27,3)$ & (x,y)

HAS SLOPE : $-\frac{x}{9y}$

$$x^2 + 9y^2 = 81$$

SLOPE = $\frac{\Delta y}{\Delta x} = \frac{3-y}{27-x} = \frac{-x}{9y}$

$$9y(3-y) = -x(27-x)$$

$$27y - 9y^2 = -27x + x^2$$

$$27(x+y) = x^2 + 9y^2 = 81$$

$$x + y = 3 \longrightarrow y = 3 - x$$

$x^2 + 9y^2 = 81$

$x^2 + 9(3-x)^2 = 81$

$x^2 + 9(9 - 6x + x^2) = 81$

$x^2 + \cancel{81} - 54x + 9x^2 = \cancel{81}$

$10x^2 - 54x = 0$

$x(10x - 54) = 0$

$x = 0$

$y = 3 - 0 = 3$

$x = \frac{54}{10} = 5.4$

$y = 3 - 5.4 = -2.4$

$(0, 3)$

$(5.4, -2.4)$

EQ OF LINE

$y - 3 = \frac{dy}{dx}(x - 27)$

$-\frac{5.4}{9(-2.4)} = .25$

$y - 3 = .25(x - 27)$

$y = .25x - 3.75$

Find dy/dx by implicit differentiation.

$$\sqrt{7xy} = 1 + x^2y$$

$y' =$ $\boxed{\dfrac{4xy\sqrt{5xy} - 5y}{5x - 2x^2\sqrt{5xy}}}$

$$\frac{d}{dx}\left[(7xy)^{1/2}\right] = \frac{d}{dx}\left[1 + x^2 y\right]$$

$$\frac{1}{2}(7xy)^{-1/2}\frac{d}{dx}[7xy] = 0 + \frac{d}{dx}[x^2]\,y + x^2\frac{d}{dx}[y]$$

$$\frac{7}{2}(7xy)^{-1/2}\left(\frac{d}{dx}[x]\,y + x\frac{d}{dx}[y]\right) = 2xy + x^2\frac{dy}{dx}$$

$$\frac{7}{2}(7xy)^{-1/2}\left(y + x\frac{dy}{dx}\right) = 2xy + x^2\frac{dy}{dx}$$

$$\frac{7}{2}(7xy)^{-1/2}y + \frac{7}{2}(7xy)^{-1/2}x\frac{dy}{dx} = 2xy + x^2\frac{dy}{dx}$$

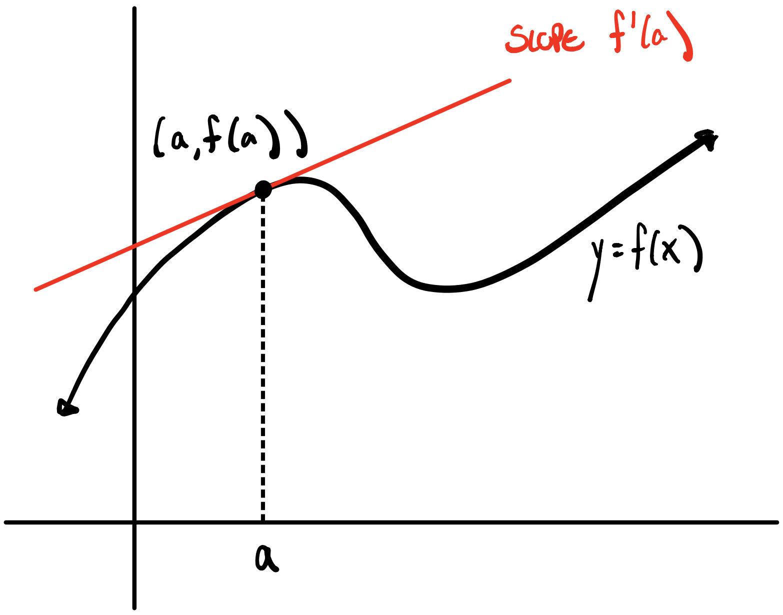
$$\frac{7}{2}(7xy)^{-1/2}x\frac{dy}{dx} - x^2\frac{dy}{dx} = 2xy - \frac{7}{2}(7xy)^{-1/2}y$$

$$\frac{dy}{dx} = \frac{2xy - \frac{7}{2}(7xy)^{-1/2}y}{\frac{7}{2}(7xy)^{-1/2}x - x^2} \cdot \frac{2(7xy)^{1/2}}{2(7xy)^{1/2}}$$

$$\frac{dy}{dx} = \frac{4xy\sqrt{7xy} - 7y}{7x - 2x^2\sqrt{7xy}}$$

$$\sqrt{7xy} = 1 + x^2y$$

§2.9 Linear Approximation & Differentials



(Line thru (a,b) with slope m: $y - b = m(x-a)$)

Tangent line to $y=f(x)$ at $(a, f(a))$:

$$y - f(a) = f'(a)(x-a)$$

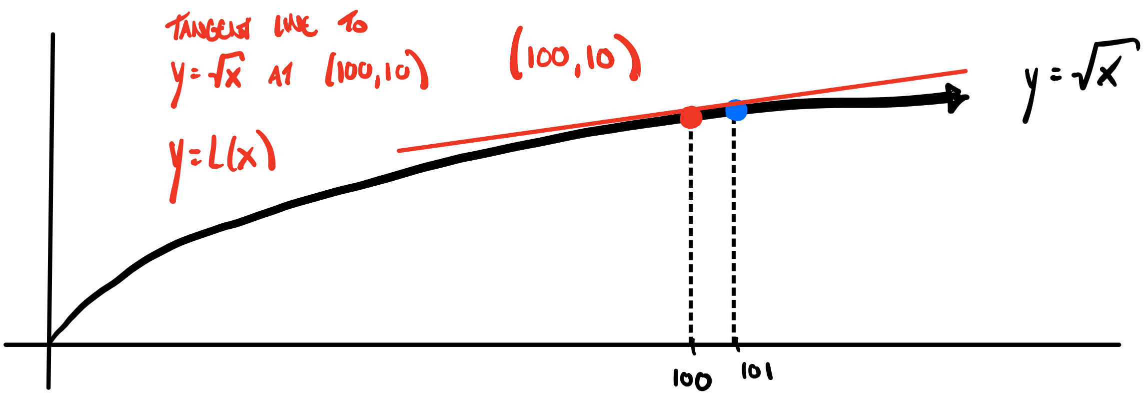
$$y = \underbrace{f(a) + f'(a)(x-a)}_{L(x)}$$

$L(x)$ is the linear function that does the best job of approximating the differentiable function $f(x)$ when input x is near a.

$L(x) = f(a) + f'(a)(x-a)$ is called the **linearization** of f at a.

$\underbrace{f(x) \approx L(x)}_{\text{LINEAR APPROXIMATION}}$ when $|x-a|$ is small.

ex. Use linear approximation to estimate $\sqrt{101}$.



Consider $f(x) = \sqrt{x}$, $f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

$y = L(x) = f(a) + f'(a)(x-a)$, $a = 100$

$f(x) = \sqrt{x}$

$$L(x) = \sqrt{100} + \frac{1}{2\sqrt{100}}(x-100)$$

$$L(x) = 10 + .05(x-100)$$

LINEAR APPROX: $f(x) \approx L(x)$ WHEN $|x-a|$ IS SMALL

$$\sqrt{101} = f(101) \approx L(101) \quad , \quad |101-100| = 1 \text{ IS SMALL}$$

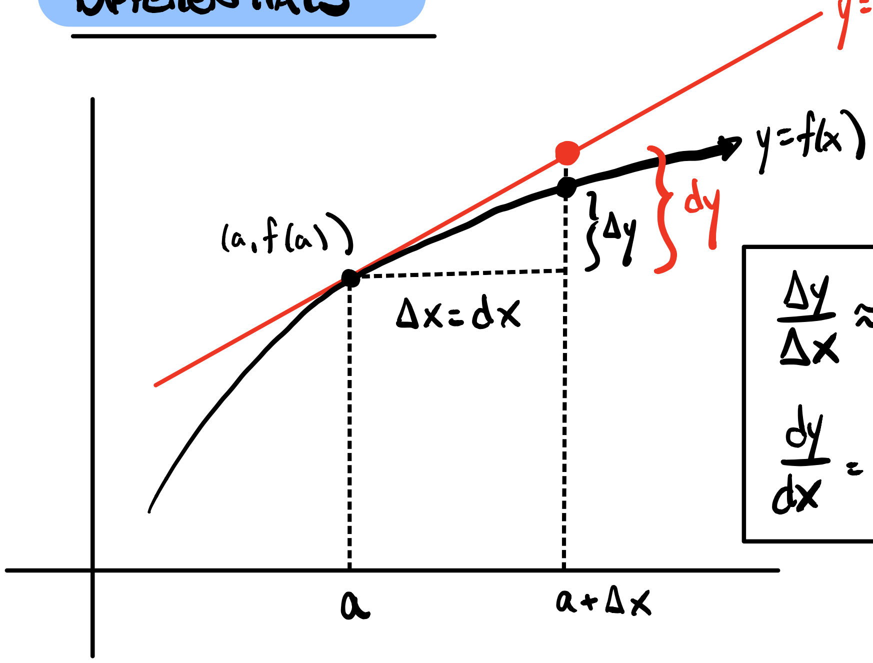
$$\approx 10 + .05\,(101-100) = 10.05$$

$$\sqrt{101} = 10.04987562\ldots$$

10.0499

DIFFERENTIALS

$y = L(x) = f(a) + f'(a)(x-a)$



$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(a)$$

$\frac{\Delta y}{\Delta x} \approx f'(a)$	:	$\Delta y \approx f'(a)\Delta x$
$\frac{dy}{dx} = f'(a)$	:	$dy = f'(a)\,dx$

$\Delta y = f(a+\Delta x) - f(a)$

$dy = L(a+\Delta x) - L(a)$

$$dy = f'(a)\,dx$$

DEPENDENT DIFFERENTIAL VARIABLE

INDEPENDENT DIFFERENTIAL VARIABLE

$dy = L(a) - L(x)$

CHANGE IN OUTPUT OF THE LINEARIZATION OF f

$dx = x - a$

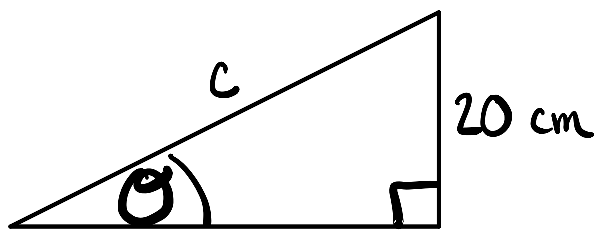
CHANGE IN INPUT

OF f.

One side of a right triangle is known to be 20 cm long and the opposite angle is measured as 30°, with a possible error of ±1°.

(a) Use differentials to estimate the error in computing the length of the hypotenuse.

(b) What is the percentage error?



$$\sin\theta = \frac{20}{c}$$

$$c = \frac{20}{\sin\theta} = 20\csc\theta$$

$$\frac{dc}{d\theta} = -20\csc\theta\cot\theta$$

DIFFERENTIALS: $dc = -20\csc\theta\cot\theta\, d\theta$

$$d\theta = \pm 1^\circ \times \frac{\pi}{180} = \pm\frac{\pi}{180}$$

$$\theta = 30^\circ = 30^\circ \times \frac{\pi}{180^\circ} = \frac{\pi}{6} \text{ RADIANS}$$

$$dc = -20\csc\frac{\pi}{6}\cot\frac{\pi}{6}\left(\pm\frac{\pi}{180}\right)$$

$$dc = -20(2)\sqrt{3}\left(\pm\frac{\pi}{180}\right) = \pm\frac{40\sqrt{3}\,\pi}{180} \approx \pm 1.2 \text{ cm}$$

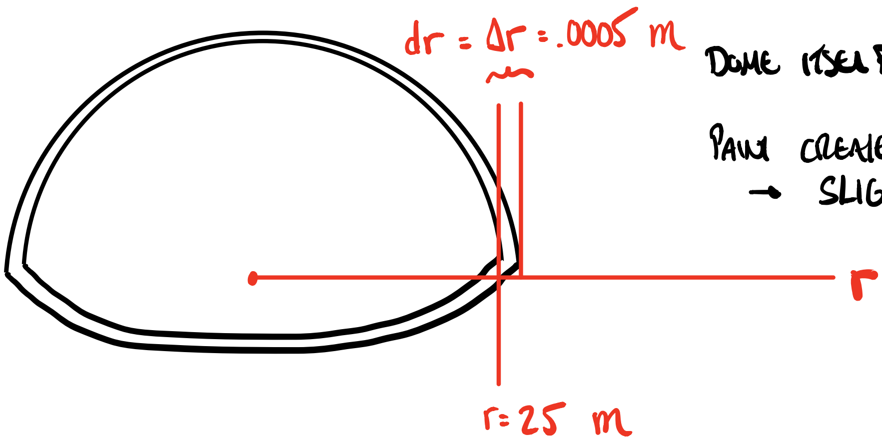
ABSOLUTE ERROR $\approx$ 1.2 cm

$$\text{RELATIVE ERROR} := \frac{\text{ABS. ERROR}}{\text{VALUE}} = \frac{1.2 \text{ cm}}{40} = .03$$

$\pm 3\%$

$$c = 20\csc\theta = 20\csc\left(\frac{\pi}{6}\right) = 40$$

Use differentials to estimate the amount of paint needed to apply a coat of paint 0.05 cm thick to a hemispherical dome with diameter 50 m.



DOME ITSELF ON INSIDE

PAINT CREATES A THIN LAYER
→ SLIGHTLY LARGER DOME

FIND dV WHEN $r = 25$ & $\Delta r = .0005 = dr$

VOLUME OF SPHERE: $V = \frac{4}{3}\pi r^3$

HEMISPHERE: $V = \frac{2}{3}\pi r^3$ $\qquad \frac{dV}{dr} = 2\pi r^2$

$$dV = 2\pi r^2 \, dr$$

$$dV = 2\pi (25)^2 (.0005) = 1.96 \text{ m}^3$$

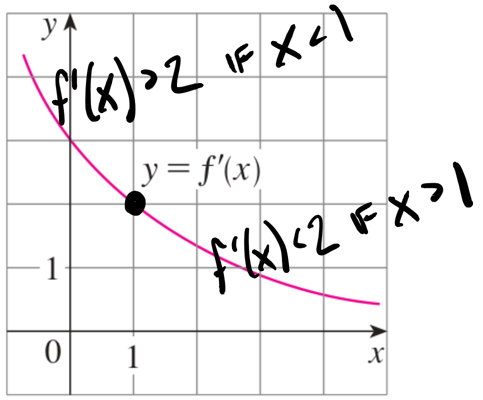
Suppose that the only information we have about a function f is that $f(1) = 5$ and the graph of its *derivative* is as shown.

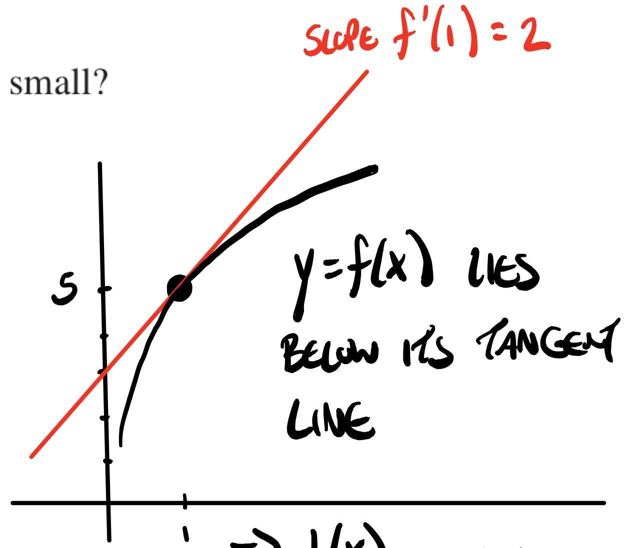
(a) Use a linear approximation to estimate $f(0.9)$ and $f(1.1)$.

(b) Are your estimates in part (a) too large or too small? Explain.

SKETCH $y = f(x)$

f' IS DECREASING





$y = f(x)$ LIES BELOW ITS TANGENT LINE

$\Rightarrow L(x)$ IS AN OVERESTIMATE

(a) $f(x) \approx L(x) = f(a) + f'(a)(x-a)$, $a = 1$

$$= 5 + 2(x-1)$$
$$= 3 + 2x$$

$f(0.9) \approx L(0.9) = 3 + 2(.9) = 4.8$

$f(1.1) \approx L(1.1) = 3 + 2(1.1) = 5.2$