§2.2 The Derivative as a Function

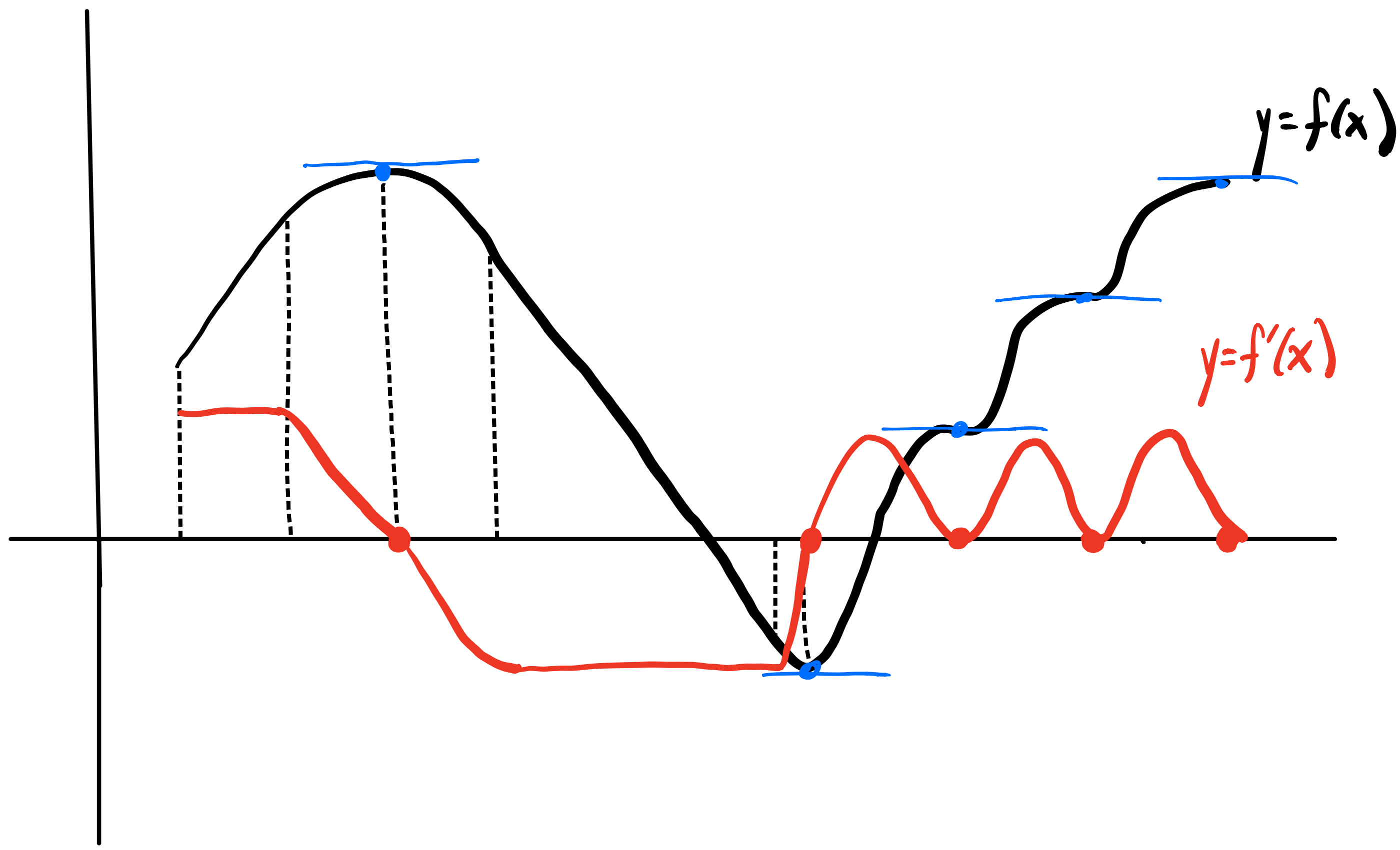
Def: Given a function $f(x)$,
the function $f'(x)$ is defined to be

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

at every point x in $\mathrm{Dom}(f)$ such that this limit exists.

<u>ex.</u> $f(x) = x^2$: $f'(a) \to 2a$ substitute $a = x$.
$f'(x) = 2x$

<u>ex.</u> Given the graph $y = f(x)$, sketch $y = f'(x)$



Note: $\mathrm{Dom}(f') \subseteq \mathrm{Dom}(f)$

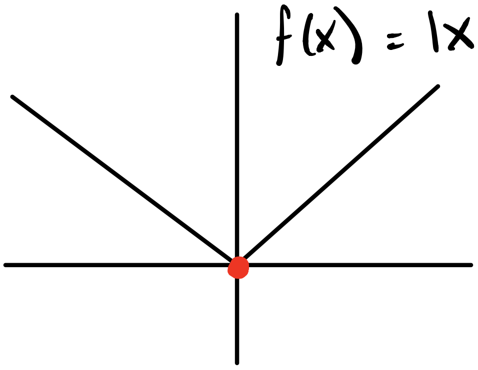
"SUBSET"

$\Rightarrow$ COULD BE POINTS WHERE $f(x)$ IS DEFINED &

$f'(x)$ IS NOT DEFINED.

How?

$f(x) = |x| = \begin{cases} x & \text{IF } x \geq 0 \\ -x & \text{IF } x < 0 \end{cases}$



CORNER

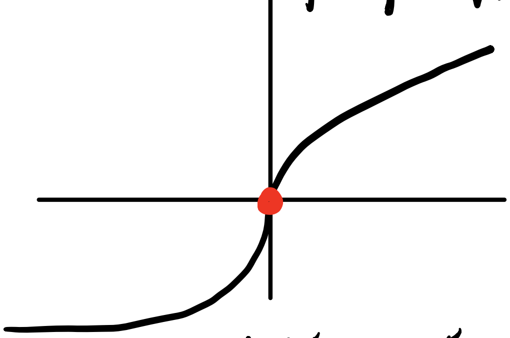
$f'(0)$ DNE

$$\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \frac{h - 0}{h} = 1$$

$$\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \frac{-h}{h} = -1$$

$$\therefore \underbrace{\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}}_{f'(0) \text{ D.N.E.}} \quad \text{D.N.E.}$$

$f(x) = \sqrt[3]{x}$

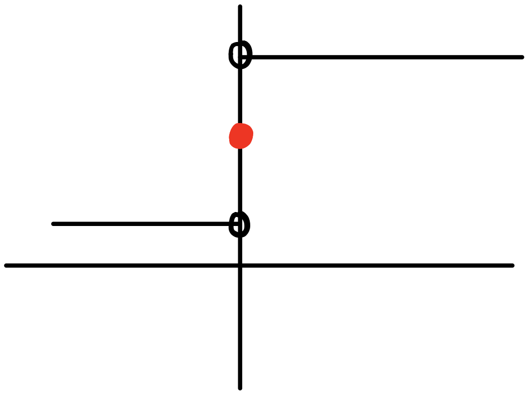


$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \infty \quad (\text{D.N.E.})$$

VERTICAL TANGENT

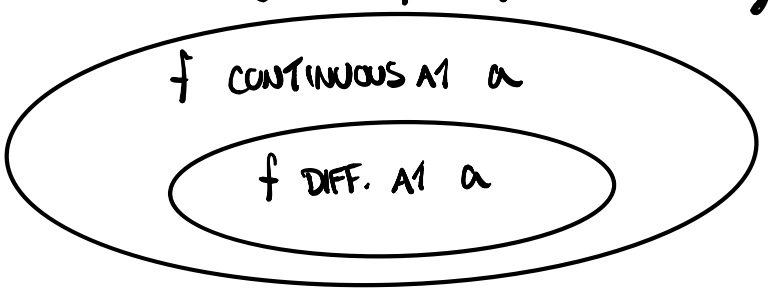
$f'(0)$ D.N.E.



$$f(x) = \begin{cases} 3 & \text{IF } x > 0 \\ 2 & \text{IF } x = 0 \\ 1 & \text{IF } x < 0 \end{cases}$$

$f'(0)$ D.N.E.

IN FACT: IF f IS NOT CONTINUOUS AT a
THEN f IS NOT DIFFERENTIABLE AT a
(i.e. $f'(a)$ D.N.E.)



EQUIVALENT

THEOREM: IF f IS DIFFERENTIABLE AT a
THEN f IS CONTINUOUS AT a.

PROOF: WE MUST SHOW $\lim_{x\to a} f(x) = f(a)$

i.e. $\lim_{x\to a} \left[f(x) - f(a)\right] = 0$.

WE HAVE

$$\lim_{x\to a}\left[f(x)-f(a)\right] = \lim_{x\to a}\left[\frac{f(x)-f(a)}{x-a}(x-a)\right]$$

$$= \lim_{x\to a}\frac{f(x)-f(a)}{x-a} \cdot \lim_{x\to a}(x-a)$$

$= f'(a) \cdot 0 = 0.$ $\square$

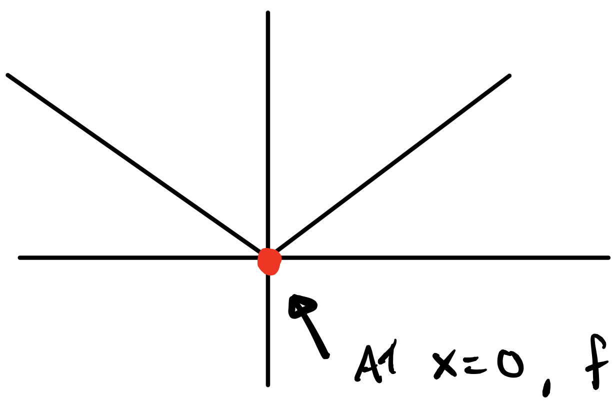
A FUNCTION BEING DIFFERENTIABLE (HAVING A DERIVATIVE) AT A POINT IS A STRONGER PROPERTY THAN SIMPLY BEING CONTINUOUS AT A POINT.

f IS CONTINUOUS AT a: GRAPH HAS NO HOLES, NO JUMPS, NO ASYMPTOTES

"CONNECTED"

f IS DIFFERENTIABLE AT a: AS WE ZOOM IN, THE GRAPH APPEARS TO BE MORE & MORE LIKE A STRAIGHT LINE WITH A DEFINED SLOPE.

"SMOOTH"



$f(x) = |x|$ IS CONTINUOUS ON $(-\infty, \infty)$ & DIFFERENTIABLE ON $(-\infty, 0) \cup (0, \infty)$.

AT $x = 0$, $f(x) = |x|$ IS CONTINUOUS BUT NOT DIFFERENTIABLE

* NOTE: THERE ARE FUNCTIONS THAT ARE CONTINUOUS ON $(-\infty, \infty)$, BUT ARE NOT DIFFERENTIABLE ANYWHERE.

RECALL: $f(x) = \begin{cases} 0 & \text{IF } x \text{ IS RATIONAL} \\ 1 & \text{IF } x \text{ IS IRRATIONAL} \end{cases}$

Other Notations

$$f'(x) = \lim_{a \to x} \frac{f(a) - f(x)}{a - x} \quad \frac{\leftarrow \Delta \text{ output}}{\leftarrow \Delta \text{ input}} = \frac{\Delta y}{\Delta x}$$

$\Delta x = a - x$: $\Delta x \to 0$ as $a \to x$

Let f be a differentiable function of x.

Let $y = f(x)$. The following are equivalent

This is NOT a fraction. It is a symbol.

$$f'(x) = y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x)$$

"The derivative of"

(an operator on the function f)

$$= D_x f(x) = D f(x)$$

And

$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a} = \left.\frac{dy}{dx}\right]_{x=a}$$

Vertical bar = "evaluate at"

Higher Order Derivatives

Function $f(x)$. Let $y = f(x)$.

(First) derivative $f'(x) = y' = \frac{d}{dx} f = \frac{dy}{dx}$

Second derivative = derivative of derivative

$$f''(x) = (f'(x))' = y'' = \frac{d}{dx}\left(\frac{d}{dx} f\right)$$

$$= \frac{d^2}{dx^2} f = \frac{d^2 y}{dx^2}$$

YOU CAN KEEP GOING :

$$f'''(x) = \frac{d^3}{dx^3} f = \frac{d^3 y}{dx^3} = y'''$$

n^{TH} DERIVATIVES , $n \geq 4$

$f''''(x)$ ✗

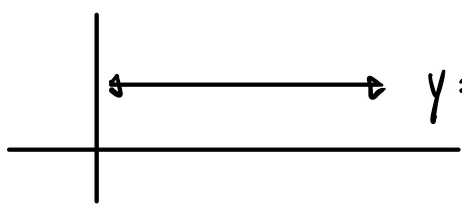
$f^{(4)}(x)$ ✓ (USUALLY STATED)

$\frac{d^4 y}{dx^4}$ ✓

§2.3 DIFFERENTIATION FORMULAS "SHORTCUTS"

THE PROCESS OF TAKING DERIVATIVES.

1. $\frac{d}{dx} c = 0$



$y = c$ (SLOPE 0)

LET $f(x) = c$, c IS CONSTANT

$f'(x) = \frac{d}{dx} f(x) = \frac{d}{dx} c$:

$$\hookrightarrow \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{c-c}{h} = \lim_{h\to 0} 0 = 0.$$

2. Power Rule : $\frac{d}{dx} x^n = n x^{n-1}$

ex. $f(x) = x^2$, $f'(x) = 2x^{2-1} = 2x$

$g(x) = x^5$, $f'(x) = 5x^4$

Proof: Let $f(x) = x^n$

Then $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$$

Binomial Formula :

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$$

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$\begin{array}{c} 1 \\ 1 \quad 1 \\ 1 \quad 2 \quad 1 \\ 1 \quad 3 \quad 3 \quad 1 \\ 1 \quad 4 \quad 6 \quad 4 \quad 1 \\ 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\ 1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\ \vdots \end{array}$$

$$(a+b)^6 = 1a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + 1b^6$$

$$= \lim_{h\to 0} \frac{(\cancel{x^n} + nx^{n-1}h + h^2(\ldots)) - \cancel{x^n}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}\left(nx^{n-1} + h(\dots)\right)}{\cancel{h}}$$

$$= \lim_{h \to 0} nx^{n-1} + \underbrace{\lim_{h \to 0} h(\overbrace{\quad\dots\quad}^{\text{POLYNOMIAL}})}_{} \qquad \text{PLUG IN}$$

$$nx^{n-1} + 0 \qquad \checkmark$$