§8.3 Conditional Probability, Intersection, and Idependence

Example 1. A car insurance company kept detailed records on all of its 100 million policy holders during 2019. The records show that most of their policy holders obeyed traffic laws and drove safe. In fact, during 2019, only 2% of policy holders received points on their license for a moving violation, 1% of policy holders got into a major accident, and only 0.5% of policy holders both received points on their license for a moving violation and got into a major accident.

- What is the probability that a policy holder gets into a major accident?
- What is the probability that a policy holder with points on their license gets into a major accident?
- What is the probability that a policy holder who gets into a major accident has points on their license?

DEFINITION Conditional Probability

For events A and B in an arbitrary sample space S, we define the **conditional probability of A given B** by

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \qquad P(B) \neq 0 \tag{1}$$

Example 2. If two dice are rolled, find the probability that at least one 1 is rolled given that the sum is less than or equal to 4.

THEOREM 1 Product Rule

For events A and B with nonzero probabilities in a sample space S,

$$P(A \cap B) = P(A)P(B|A) = P(B)P(A|B) \tag{2}$$

and we can use either $P(A)P(B|A)$ or $P(B)P(A|B)$ to compute $P(A \cap B)$.

Example 3. Suppose 50% if customers at a restaurant order an appetizer, and 30% of customers who order an appetizer order a dessert. What proportion of customers order both an appetizer and a dessert? What proportion of customers order an appetizer and do not order dessert?

Example 4. A jar contains 8 red marbles and 6 blue marbles. If two marbles are drawn *without replacement*, find the probability that both marbles are the same color.
Note that there are 4 events under cosideration, R_1, R_2, B_1, B_2. First answer this question using a tree diagram and the Product rule, then again using basic counting principles.

PROCEDURE Constructing Probability Trees

Step 1 Draw a tree diagram corresponding to all combined outcomes of the sequence of experiments.

Step 2 Assign a probability to each tree branch. (This is the probability of the occurrence of the event on the right end of the branch subject to the occurrence of all events on the path leading to the event on the right end of the branch. The probability of the occurrence of a combined outcome that corresponds to a path through the tree is the product of all branch probabilities on the path.*)

Step 3 Use the results in Steps 1 and 2 to answer various questions related to the sequence of experiments as a whole.

*If we form a sample space S such that each simple event in S corresponds to one path through the tree, and if the probability assigned to each simple event in S is the product of the branch probabilities on the corresponding path, then it can be shown that this is not only an acceptable assignment (all probabilities for the simple events in S are nonnegative and their sum is 1), but it is the only assignment consistent with the method used to assign branch probabilities within the tree.

DEFINITION Independence

If A and B are any events in a sample space S, we say that ***A* and *B* are independent** if

$$P(A \cap B) = P(A)P(B) \tag{3}$$

Otherwise, A and B are said to be **dependent**.

From the definition of independence one can prove (see Problems 75 and 76, Exercises 8.3) the following theorem:

THEOREM 2 On Independence

If A and B are independent events with nonzero probabilities in a sample space S, then

$$P(A|B) = P(A) \quad \text{and} \quad P(B|A) = P(B) \tag{4}$$

If either equation in (4) holds, then A and B are independent.

Example 5. A jar contains 8 red marbles and 6 blue marbles. If two marbles are drawn *with replacement*, find the probability that both marbles are the same color.

Example 6. Fill out the folloiwng probability table once such that the events A and B are independent, and again such that A and B are not independent.

	A	A'
B		
B'		

Example 7. Suppose a baseball team is heading to the playoffs.

- If they play team A the probability of winning is .8.

- If they play team B the probability of winning is .5.
- If they play team C the probability of winning is .1.

The probability that they play team A is .4, team B is .3, and team C is .3. What is the probability that this team wins against team A? That they win?

Example 8. Ann and Barbara are lpaying a tennis match. The first player to win 2 sets wins the match. For any given set, the probability that Ann wins is 2/3. Find the probability that Ann wins the match. That 3 sets are played. That the player who wins the first set goe on to win the match.

DEFINITION Independent Set of Events

A set of events is said to be **independent** if for each finite subset $\{E_1, E_2, \ldots, E_k\}$

$$P(E_1 \cap E_2 \cap \cdots \cap E_k) = P(E_1)P(E_2) \cdot \cdots \cdot P(E_k) \tag{5}$$

The next example makes direct use of this definition.

Example 9. Suppose a basketball player makes 80% of all free throws. If they take 4 free throws, find the probability that they make all 4. Make 3 out of 4.