MODULE 7 : CONDITIONAL PROBABILITY

SUMMARY Key Concepts

Conditional Probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \qquad P(B|A) = \frac{P(B \cap A)}{P(A)}$$

Note: $P(A|B)$ is a probability based on the new sample space B, while $P(A \cap B)$ and $P(B)$ are probabilities based on the original sample space S.

Product Rule

$$P(A \cap B) = P(A)P(B|A) = P(B)P(A|B)$$

Independent Events

- A and B are **independent** if

$$P(A \cap B) = P(A)P(B)$$

- If A and B are independent events with nonzero probabilities, then

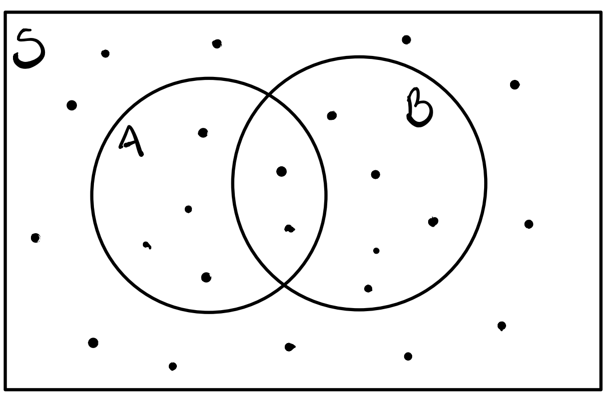
$$P(A|B) = P(A) \quad \text{and} \quad P(B|A) = P(B)$$

- If A and B are events with nonzero probabilities and either $P(A|B) = P(A)$ or $P(B|A) = P(B)$, then A and B are independent.
- If $E_1, E_2, \ldots, E_n$ are independent, then

$$P(E_1 \cap E_2 \cap \cdots \cap E_n) = P(E_1)P(E_2) \cdot \cdots \cdot P(E_n)$$

CONDITIONAL PROBABILITY

TWO EVENTS $A, B \subseteq S$



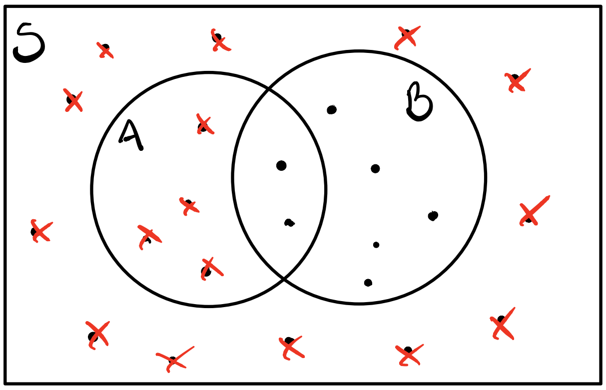
SIZE OF SAMPLE SPACE IS FINITE.

ALL POSSIBLE OUTCOMES ARE EQUALLY LIKELY.

$$P(A) = \frac{n(A)}{n(S)} = \frac{6}{23}$$

NOW SUPPOSE YOU HAVE KNOWLEDGE THAT B HAS OCCURRED.

NOW WHAT IS THE PROB. OF A?



UPDATE SAMPLE SPACE

ELIMINATE SOME POSSIBLE OUTCOMES (THOSE NOT INSIDE OF B)

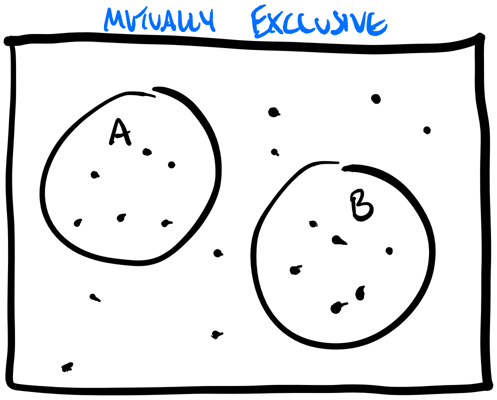
$$P(A|B) = \frac{n(A \cap B)}{n(B)} = \frac{2}{7}$$

$$P(A|B) = \frac{n(A \cap B)/n(S)}{n(B)/n(S)} \begin{array}{l} \leftarrow P(A \cap B) \\ \leftarrow P(B) \end{array}$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

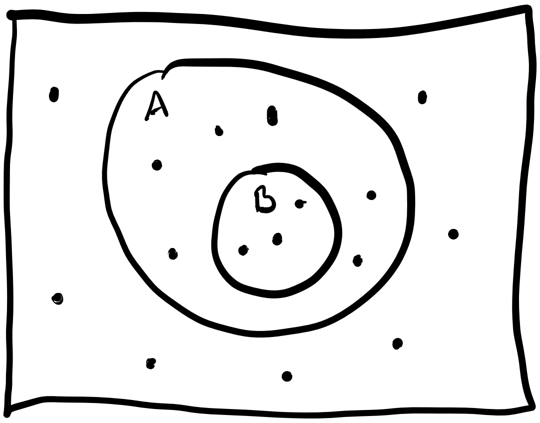
EXTREME CASES:

MUTUALLY EXCLUSIVE



$$P(A) = \frac{6}{20}$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = 0$$



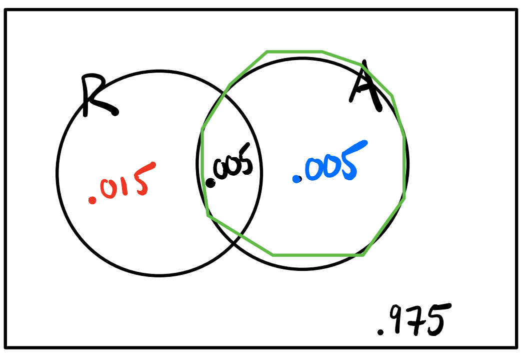
$$P(A|B) = 1$$

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$$

Example. A car insurance company kept detailed records on all of its 8.2 million policy holders during 2019. The records show that most of their policy holders obeyed traffic laws and drove safe. In fact, during 2019, only 2% of policy holders received points on their license for a moving violation, 1% of policy holders got into a major accident, and only 0.5% of policy holders both received points on their license for a moving violation and got into a major accident.

1. What is the probability that a policy holder gets into a major accident?
2. What is the probability that a policy holder with points on their license gets into a major accident?
3. What is the probability that a policy holder who gets into a major accident has points on their license?

UPDATED SAMPLE SPACE



R = RECEIVED POINTS ON LICENSE

A = ACCIDENT

$P(R) = .02$

$\rightarrow P(A) = .01$

$P(A\cap R) = .005$

$$P(A) = P(A\cap R) + P(A\cap R')$$

$$.01 = .005 + .005$$

1. $P(A) = .01$

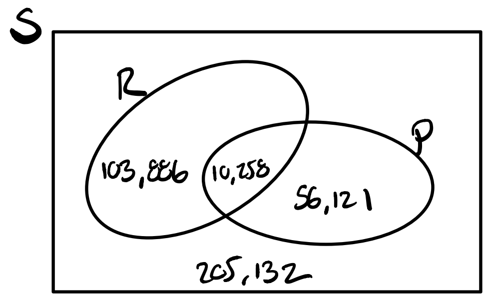
2. $P(A|R) = \dfrac{P(A\cap R)}{P(R)} = \dfrac{.005}{.02} = .25$

3. $P(R|A) = \dfrac{P(R\cap A)}{P(A)} = \dfrac{.005}{.01} = 50\%$

Example. The marketing department for a website that asks visitors to login with their Facebook account has collected data on its users from Facebook Analytics. This data is summarized in the following tables.

1. What percentage of your visitors follow the rock?
2. What percentage of your visitors follow Cats of Instagram?
3. If your company has budgeted \$2500 to spend on Instagram ads, at \$1.25 per click (you only pay when someone clicks the ad/visits your website), should you target your ads toward followers of The Rock or followers of Cats of Instagram?

Website Visitors that...	Place an order	Do not place an order	Total
Follow The Rock	10,258	103,886	114,144
Do not follow the Rock	56,121	205,132	261,253
Total	66,379	309,018	375,397



Website Visitors that...	Place an order	Do not place an order	Total
Follow Cats of IG	608	1,452	2,060
Do not follow Cats of IG	65,771	307,566	373,337
Total	66,379	309,018	375,397

1. $P(R) = \dfrac{n(R)}{n(S)} = \dfrac{114,144}{375,397} = .3041$

2. $P(C) = \dfrac{n(C)}{n(S)} = \dfrac{2060}{375,397} = .0055$

3. Find $P(P|R) = \dfrac{n(P \cap R)}{n(R)} = \dfrac{10,258}{114,144} = .0900$

$P(P|C) = \dfrac{n(P \cap C)}{n(C)} = \dfrac{608}{2060} = .2951$

$P(P \mid C) > P(P \mid R) \Rightarrow$ TARGET FOLLOWERS OF CATS ON IG.

Example. You know that your neighbors have two children. Then one day you see one of the children playing outside, and this child is a boy. What is the probability that the other child is also a boy?

LET $S = \{\underbrace{BB, BG, GB, GG}_{\text{EQUALLY LIKELY}}\}$

$P(BB) = \frac{1}{4}$

$P(1B\ 1G) = P(BG \cup GB) = P(BG) + P(GB) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

GIVEN THAT ONE CHILD IS A BOY, UPDATE SAMPLE SPACE

$S = \{\underbrace{BB, BG, GB, \cancel{GG}}_{\text{EQUALLY LIKELY}}\}$

$P(BB \mid GG^c) = \frac{1}{3}$

$P(B) \times P(A \mid B) = \frac{P(A \cap B)}{\cancel{P(B)}} \times \cancel{P(B)}$

MULTIPLICATION RULE

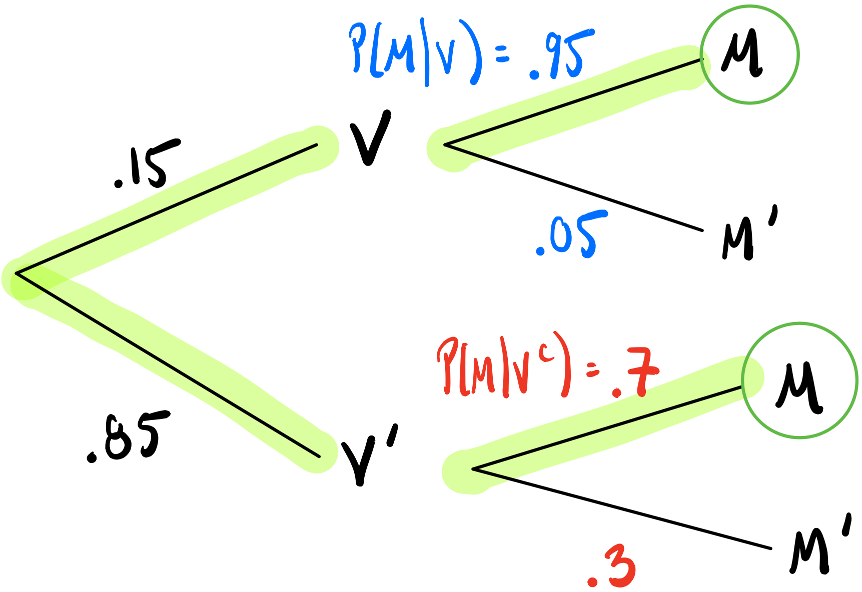
$P(A \cap B) = P(B)P(A|B) = P(A)P(B|A)$

Example. Your company is about to begin a kickstarter campaign and wants to know the probability that you will be able to raise 1 million dollars in 8 weeks. You know that if the campaign video goes viral then the probability is 95%, and if the campaign does not go viral then there is a 70% chance. Suppose there is a 15% chance that the campaign video will go viral.

CONDITIONAL CLAUSE

$\Rightarrow$ COND PROB.

1. What is the probability that the campaign video goes viral but the campaign is not able to raise 1 million dollars?
2. What is the probability that the campaign does not go viral and the campaign still raises 1 million dollars?
3. What is the probability that the campaign raises 1 million dollars?



1. $P(V \cap M') = P(V)P(M'|V) = (.15)(.05) = .0075$

2. $P(V' \cap M) = P(V')P(M|V') = (.85)(.7) = .595$

3. $P(M) = P(M \cap V) + P(M \cap V')$

$= P(V)P(M|V) + P(V')P(M|V')$

$= (.15)(.95) + (.85)(.7)$

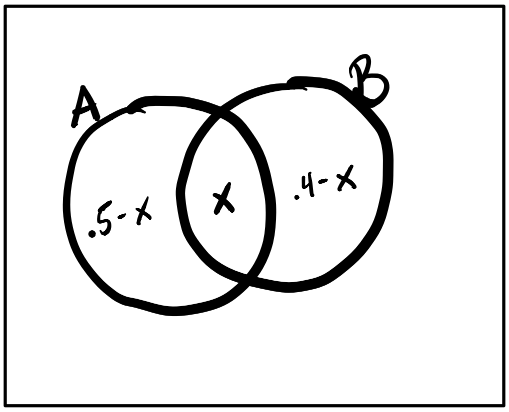
$= .7375$

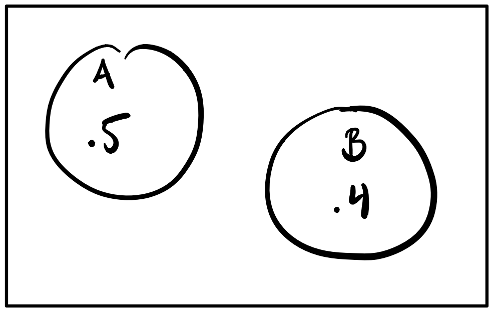
INDEPENDENT EVENTS

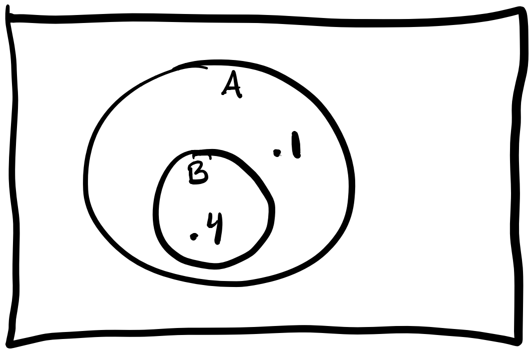
GIVEN 2 EVENTS $A, B \subseteq S$.

$P(A) = .5$, $P(B) = .4$

FIND $P(A \cap B)$.







$$P(A \cap B) = \begin{cases} P(A)\,P(B|A) \\ P(B)\,P(A|B) \end{cases}$$

↑ UNKNOWN

INDEPENDENT

$P(J) = \frac{4}{52}$ $\quad P(J|\heartsuit) = \frac{1}{13}$

SAME

IF A & B ARE INDEPENDENT

↳ DEF: $P(A|B) = P(A)$
$P(B|A) = P(B)$

IN THIS SPECIAL CASE: $P(A \cap B) = P(A)P(B)$

Bayes Formula

THEOREM 1 Bayes' Formula

Let $U_1, U_2, \ldots, U_n$ be n mutually exclusive events whose union is the sample space S. Let E be an arbitrary event in S such that $P(E) \neq 0$. Then,

$$P(U_1|E) = \frac{P(U_1 \cap E)}{P(E)} = \frac{P(U_1 \cap E)}{P(U_1 \cap E) + P(U_2 \cap E) + \cdots + P(U_n \cap E)}$$

$$= \frac{P(E|U_1)P(U_1)}{P(E|U_1)P(U_1) + P(E|U_2)P(U_2) + \cdots + P(E|U_n)P(U_n)}$$

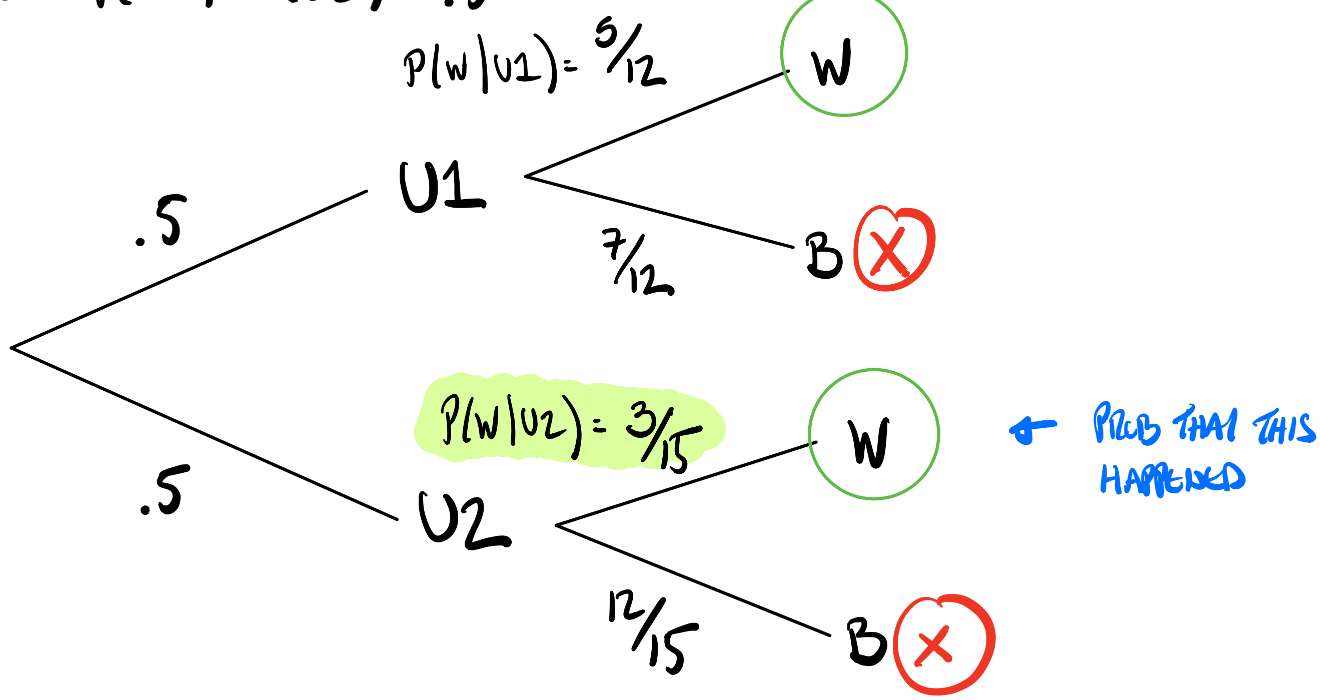
Similar results hold for $U_2, U_3, \ldots, U_n$.

Urn 1 contains 5 white balls and 7 black balls. Urn 2 contains 3 whites and 12 black. A fair coin is flipped; if it is Heads, a ball is drawn from Urn 1, and if it is Tails, a ball is drawn from Urn 2. Suppose that this experiment is done and you learn that a white ball was selected. What is the probability that this ball was in fact taken from Urn 2? (i.e., that the coin flip was Tails)

LET $U1$ = URN 1 $\quad$ W = WHITE BALL

$U2$ = URN 2 $\quad$ B = BLACK BALL

GIVEN $P(U1) = P(U2) = .5$



FIND $P(U2 \mid W) = \dfrac{P(U2 \cap W)}{P(U2 \cap W) + P(U1 \cap W)}$ ← POSSIBLE OUTCOMES THAT REMAIN.

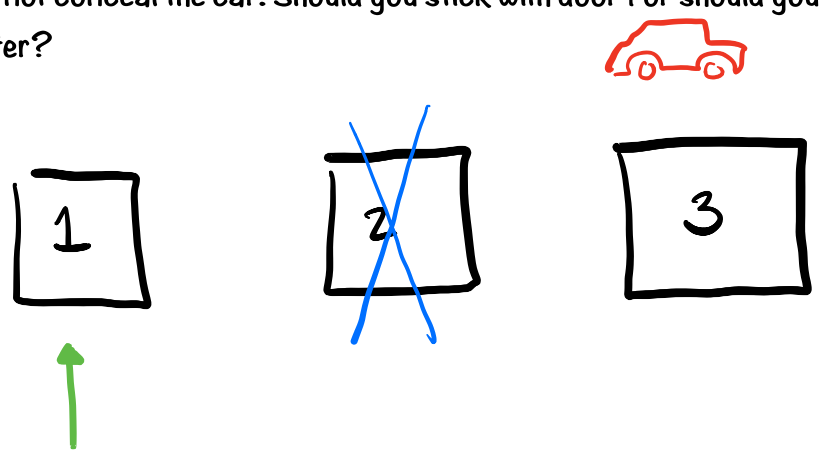
BAYES' RULE

$$= \frac{P(U2)P(W \mid U2)}{P(U2)P(W \mid U2) + P(U1)P(W \mid U1)}$$

$$= \frac{\left(\frac{1}{2}\right)\left(\frac{3}{15}\right)}{\left(\frac{1}{2}\right)\left(\frac{3}{15}\right) + \left(\frac{1}{2}\right)\left(\frac{5}{12}\right)}$$

$$= .3243$$

Example. (The Monty Hall problem) there once was a game show on TV where the winner had to open one of three closed doors, and they would win whatever was behind their chosen door. One of the doors conceals a brand new car, but the other doors conceal only a modest prize. After the winner chooses one of the doors, the host Monty Hall opens one of the other two unchosen doors to reveal that it does not conceal the car and then asks you if you want to switch your choice to the other closed door. He always does this — if one of the unchosen doors conceals the car then he opens the other one, and if neither of the unchosen doors conceals the car then he opens one of the unchosen doors at random. Suppose you are the winner of this game show and you choose door 1. Then Monty Hall opens doors 3 and shows that it does not conceal the car. Should you stick with door 1 or should you switch to door 2? Or does it not matter?

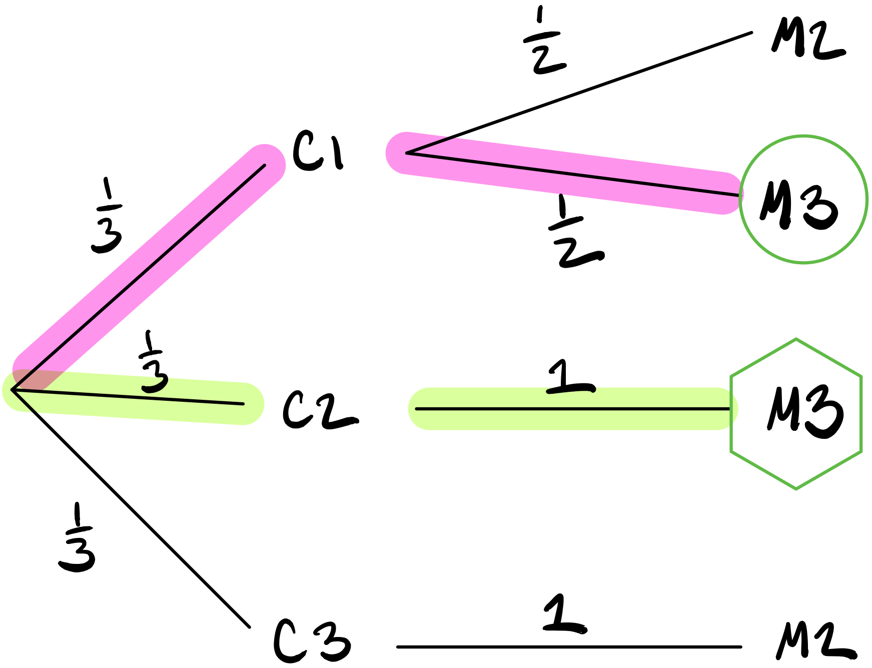


C1 : CAR IS BEHIND DOOR 1

C2 : CAR IS BEHIND DOOR 2

C3 : 3

& YOU PICK DOOR 1



FIND $P(C2 \mid M3) = \dfrac{P(C2 \cap M3)}{P(M3)}$

$$= \frac{(\frac{1}{3})(1)}{(\frac{1}{3})(1) + (\frac{1}{3})(\frac{1}{2})} = \frac{2}{3}$$