MODULE 5: COUNTING PRINCIPALS (COMBINATORICS)

THE MULTIPLICATION PRINCIPAL:

IF AN EVENT CAN BROKEN INTO k STAGES SUCH THAT

THE 1^{ST} STAGES HAS n_1 POSSIBLE OUTCOMES,

THE 2^{nd} STAGES HAS n_2 POSSIBLE OUTCOMES,

$\vdots$

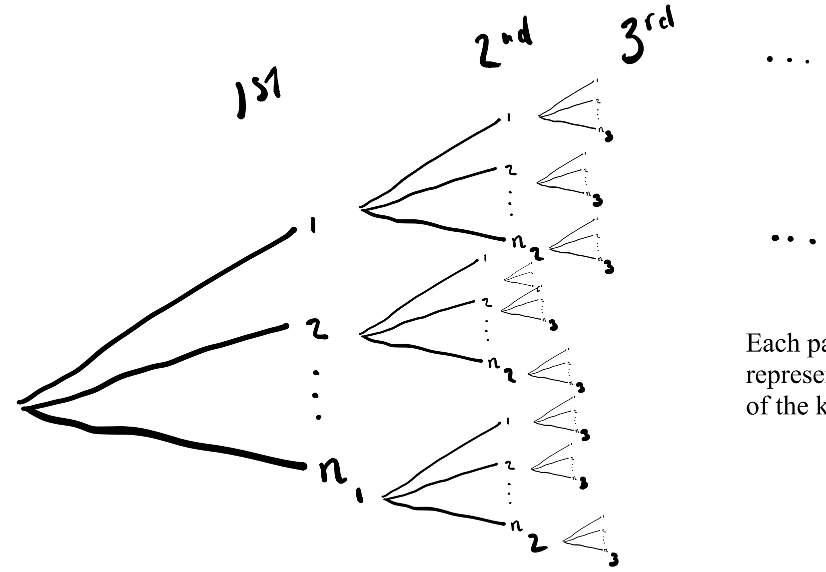
THE k^{TH} STAGES HAS n_k POSSIBLE OUTCOMES,

THEN IN TOTAL, THE EVENT HAS

$$n_1 \times n_2 \times \dots \times n_k$$

POSSIBLE OUTCOMES.

TREE DIAGRAM: ILLUSTRATION OF ALL POSSIBLE OUTCOMES



Each path through the tree diagram from left to right represents a sequence of possible outcomes, one for each of the k stages.

The total number of path through the tree from left to right is...

n_1 GROUPS OF n_2 GROUPS OF n_3 = $n_1 \times n_2 \times n_3$

Example 1.

Suppose a company requires its customers to create a PIN composed of 4 digits 0-9.

1. How many PINs are possible?
2. How many PINs are possible if successive digits must be different?
3. How many PINs are possible if each digit must be distinct?

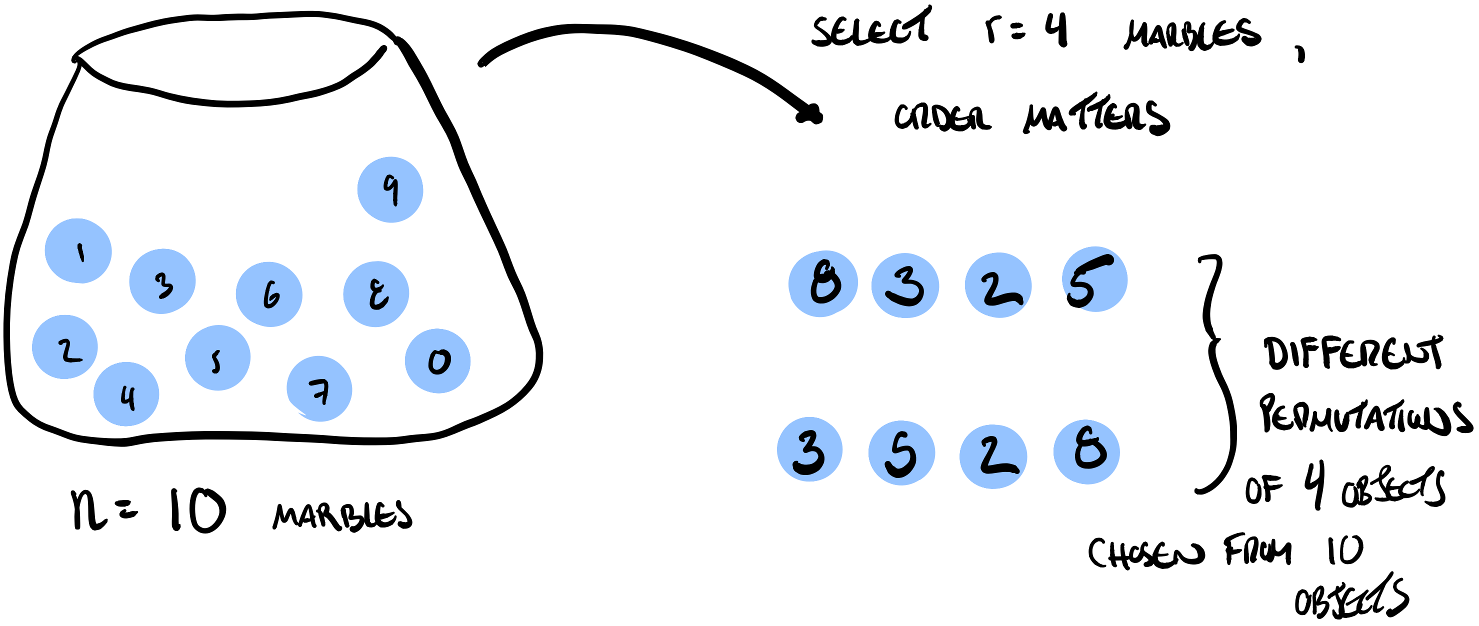
1. $\underset{1^{ST}}{\underline{10}} \times \underset{2^{nd}}{\underline{10}} \times \underset{3^{rd}}{\underline{10}} \times \underset{4^{TH}}{\underline{10}} = \underline{10{,}000}$

2. $\underset{1^{ST}}{\underline{10}} \quad \underset{2^{nd}}{\underline{9}} \quad \underset{3^{rd}}{\underline{9}} \quad \underset{4^{TH}}{\underline{9}} = \underline{7{,}200}$

   ex. 7 6 7 6

3. $\underset{1^{ST}}{\underline{10}} \quad \underset{2^{nd}}{\underline{9}} \quad \underset{3^{rd}}{\underline{8}} \quad \underset{4^{TH}}{\underline{7}} = \underline{5{,}040}$

Def: A SELECTION OF r OBJECTS, WITHOUT REPLACEMENT, TAKEN IN A SPECIFIC ORDER, FROM A COLLECTION OF n OBJECTS IS CALLED A PERMUTATION.



SELECT r = 4 MARBLES, ORDER MATTERS

n = 10 MARBLES

DIFFERENT PERMUTATIONS OF 4 OBJECTS CHOSEN FROM 10 OBJECTS

The number of permutations (order matters) of r objects selected from n objects is denoted

$${}_nP_r$$

And it is calculated as follows

$${}_nP_r = \underbrace{n(n-1)(n-2)\cdots(n-r+1)}_{r \text{ FACTORS}}\cancel{(n-r)}\cdots\cancel{(2)}\cancel{(1)}$$

NOTE: n-FACTORIAL $n! = n(n-1)(n-2)\cdots(2)(1)$

$${}_nP_r = \frac{n!}{(n-r)!}$$

Example 2.

A musician has written and recorded 22 songs. If she want to release 10 of these songs as an album (order matters!), how many possible ways are there for her to do this?

$$\underline{22} \times \underline{21} \times \underline{20} \times \underline{19} \times \underline{18} \times \underline{17} \times \underline{16} \times \underline{15} \times \underline{14} \times \underline{13}$$

COUNTING THE # OF PERMUTATIONS OF 10 OBJECTS

SELECTED FROM A COLLECTION OF 22 OBJECTS

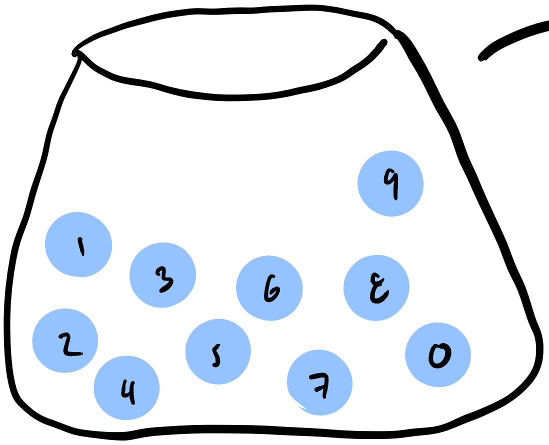
$${}_{22}P_{10} = \frac{22!}{(22-10)!} = \frac{22!}{12!} = 2.3465 \times 10^{12}$$

DEF:

A combination is a selection of r objects taken from a collection of n objects, where the order of selection does not matter. The only thing that matters is which objects are chosen.

The number of combinations of r objects taken from n objects is denoted

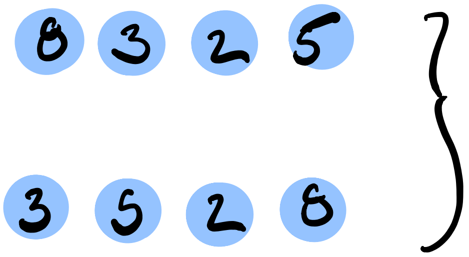
$$_nC_r$$



$n = 10$ MARBLES

SELECT $r = 4$ MARBLES,

ORDER DOES NOT MATTER



SAME COMBINATION

THESE SHOULD ONLY BE COUNTED AS ONE COMBINATION

Example 4.

How many ways are there to choose 3 aces from a standard deck of ~~52 cards~~ 4 ACES if the order doesn't matter?

Let S, C, H, and D represent the ace of spades, ace of clubs, ace of diamonds, and ace of hearts, respectively.

$_3P_3 = 3! = 6$ PERMUTATIONS CORRESPONDING TO THE SAME COMBINATION.

$_4C_3$

SCH, SHC, CSH, CHS, HSC, HCS,
CHD, CDH, HCD, HDC, DCH, DHC,
HDS, HSD, DHS, DSH, SHD, SDH,
DSC, DCS, SDC, SCD, CDS, CSD.

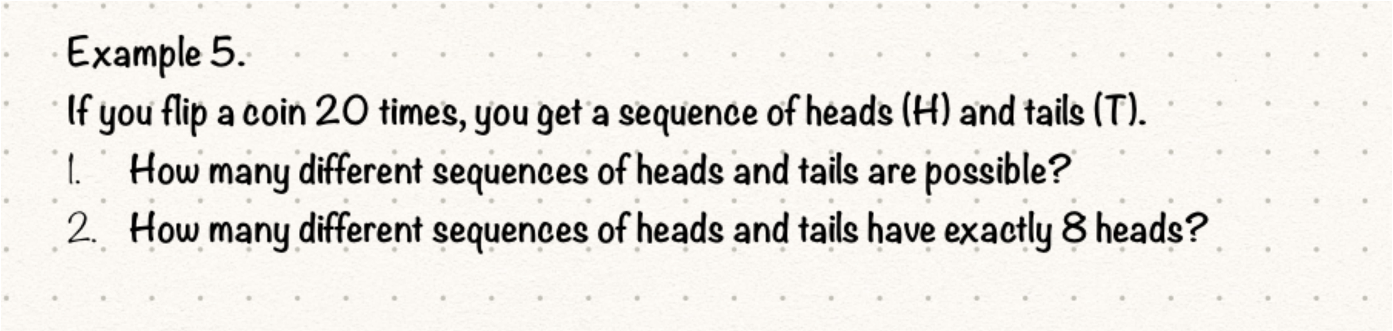
$$ {}_4P_3 = 3! \times {}_4C_3 $$

$$ \Rightarrow \quad {}_4C_3 = \frac{{}_4P_3}{3!} = \frac{24}{6} $$

$$ = 4 $$

FORMULA FOR ${}_nC_r$

$$ {}_nC_r = \frac{{}_nP_r}{r!} = \frac{n!}{(n-r)!\,r!} $$



Example 5.

If you flip a coin 20 times, you get a sequence of heads (H) and tails (T).

1. How many different sequences of heads and tails are possible?
2. How many different sequences of heads and tails have exactly 8 heads?

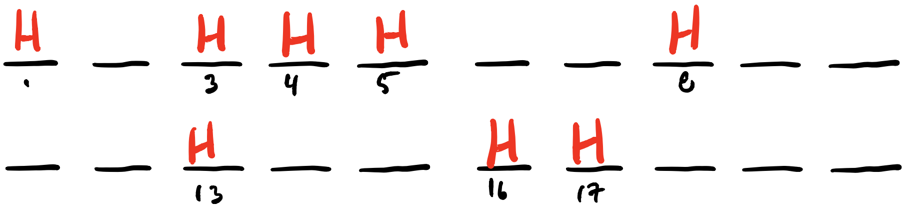
1.

$$ \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad} $$

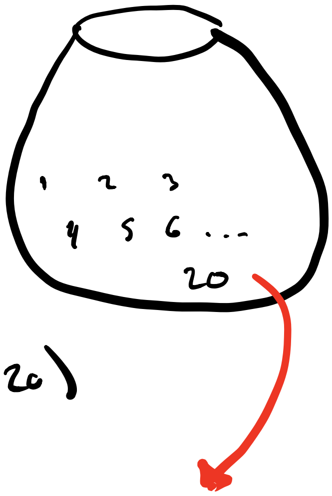
$$ \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad} $$

$$ \underbrace{2 \times 2 \times 2 \times \dots \times 2}_{\text{20 TIMES}} = 2^{20} = 1{,}048{,}576 $$

2.



EXACTLY 8 HEADS



(1) CHOOSE THE 8 POSITIONS IN THE SEQ. (OUT OF 20) WHERE THE HEADS OCCUR.

13 1 3 4 17
5 8 16

↳ # WAYS TO DO THIS:

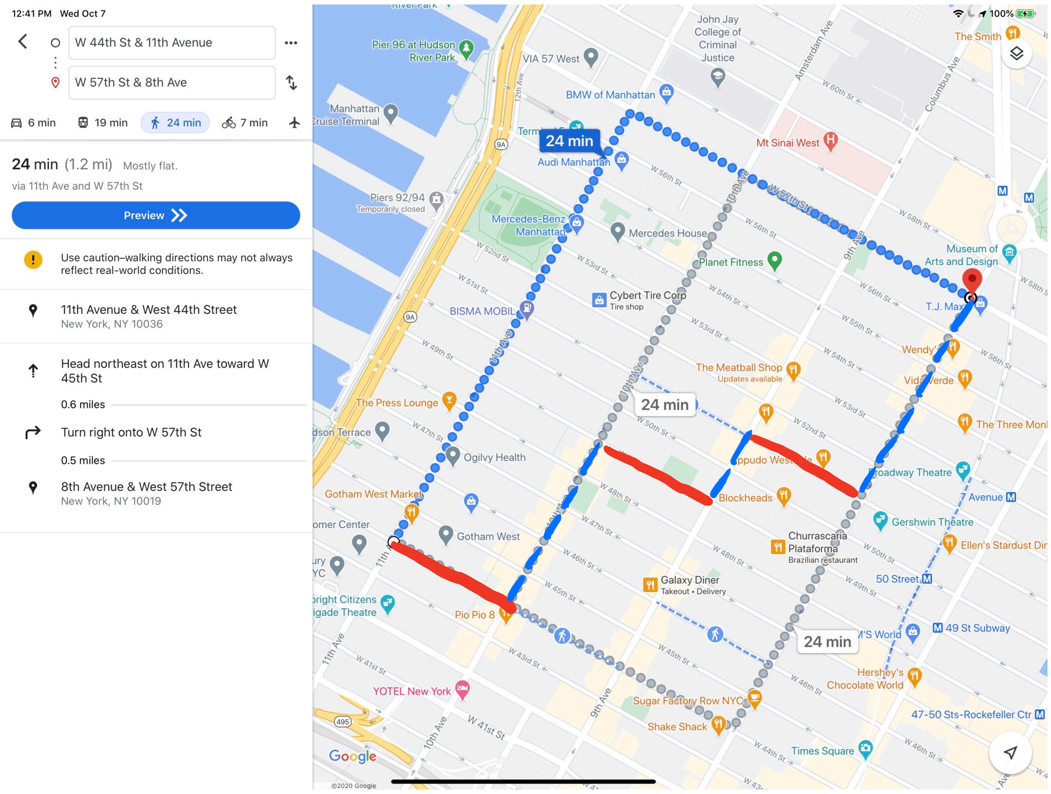
$${}_{20}C_8 = \frac{20!}{8!(20-8)!} = 125970.$$

(2) PUT TAILS IN ALL OTHER POSITIONS

TOTAL # $${}_{20}C_8 \times 1 = 125{,}970.$$

Example 6.

How many ways are there to walk from corner of W 44 St and 11th Ave to the corner of W 57 St and 8th Ave, assuming you do not go out of your way (only walk north and east)?



12:41 PM Wed Oct 7

W 44th St & 11th Avenue

W 57th St & 8th Ave

6 min 19 min 24 min 7 min

24 min (1.2 mi) Mostly flat.

via 11th Ave and W 57th St

Preview

Use caution–walking directions may not always reflect real-world conditions.

11th Avenue & West 44th Street
New York, NY 10036

Head northeast on 11th Ave toward W 45th St

0.6 miles

Turn right onto W 57th St

0.5 miles

8th Avenue & West 57th Street
New York, NY 10019

WALK NORTH 13 BLOCKS

WALK EAST 3 BLOCKS

A POSSIBLE ROUTE IS A SEQUENCE OF 13 NORTH & 3 EASTS

E N N N N N E N N E N N N N N N

N E N N N N E N N N N N N E N N

⋮

WALK 16 BLOCK: 13 NORTH, 3 EAST.

CHOOSE 3 (OUT OF 16) BLOCKS TO WALK EAST $_{16}C_3$

EQUIVALENTLY:

CHOOSE 13 (OUT OF 16) BLOCKS TO WALK NORTH $_{16}C_{13}$

$$_{16}C_3 = {}_{16}C_{13} = \frac{16!}{3!\,13!} = 560$$

Example 7.

The fish section of a pet store is stocked with 8 guppies, 6 angelfish, 13 goldfish, and 9 rainbowfish.

1. How many ways are there to select one of each?
2. How many ways are there to select two of each?
3. How many ways are there to select 7 fish, if you must select at least one of each?

1.

4 STAGE EVENT	# WAYS TO DO IT
(1) CHOOSE GUPPIE	8 = ${}_8C_1$
(2) CHOOSE ANGELFISH	6 = ${}_6C_1$
(3) CHOOSE GOLDFISH	13 = ${}_{13}C_1$
(4) CHOOSE RAINBOW FISH	9 = ${}_9C_1$

$${}_8C_1 \times {}_6C_1 \times {}_{13}C_1 \times {}_9C_1$$

$$8 \times 6 \times 13 \times 9$$

$$\boxed{5616}$$

2.

4 STAGE EVENT	# WAYS TO DO IT
(1) CHOOSE GUPPIES	${}_8C_2 = 28$
(2) CHOOSE ANGELFISH	${}_6C_2 = 15$
(3) CHOOSE GOLDFISH	${}_{13}C_2 = 78$
(4) CHOOSE RAINBOW FISH	${}_9C_2 = \frac{{}_9P_2}{2!} = \frac{9\cdot\overset{4}{\cancel{8}}}{\cancel{2}\cdot 1}$ $= 36$

$$28 \times 15 \times 78 \times 36 = 1{,}179{,}360$$

3.

5 STAGE EVENT	# WAYS TO DO IT
(1) CHOOSE GUPPIE	$8 = {}_8C_1$
(2) CHOOSE ANGELFISH	$6 = {}_6C_1$
(3) CHOOSE GOLDFISH	$13 = {}_{13}C_1$
(4) CHOOSE RAINBOW FISH	$9 = {}_9C_1$
(5) CHOOSE 3 MORE FISH FROM 32 REMAINING FISH	$4960 = {}_{32}C_3$

$${}_{8}C_{1} \times {}_{6}C_{1} \times {}_{13}C_{1} \times {}_{9}C_{1} \times {}_{32}C_{1}$$

$$8 \times 6 \times 13 \times 9 \times 4960$$

$$27{,}855{,}360$$

CHALLENGE:

Example 9.

Suppose you want to give each of your 4 nieces/nephews a Christmas card containing some money. You have ten (identical) $20 bills that you want to distribute between the 4 (distinct) envelopes so that no envelope contains $0. How many ways are there to do this?

https://en.wikipedia.org/wiki/Stars_and_bars_(combinatorics)