THEOREM 2 Multiplication Principle (for Counting)

1. If two operations O_1 and O_2 are performed in order, with N_1 possible outcomes for the first operation and N_2 possible outcomes for the second operation, then there are

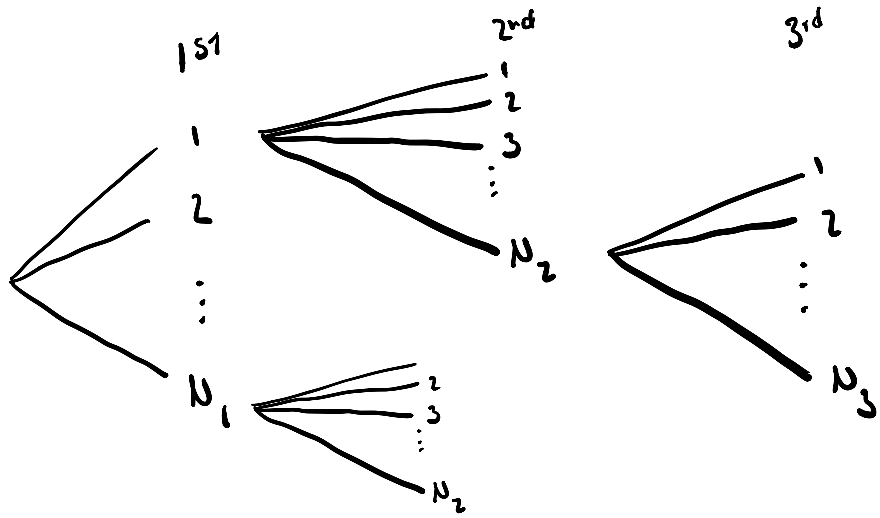
$$N_1 \cdot N_2$$

possible combined outcomes of the first operation followed by the second.

2. In general, if n operations $O_1, O_2, \ldots, O_n$ are performed in order, with possible number of outcomes $N_1, N_2, \ldots, N_n$, respectively, then there are

$$N_1 \cdot N_2 \cdot \cdots \cdot N_n$$

possible combined outcomes of the operations performed in the given order.



Example 1.

Suppose a company requires its customers to create a PIN composed of 4 digits 0-9.

1. How many PINs are possible?
2. How many PINs are possible if successive digits must be different?
3. How many PINs are possible if each digit must be distinct?

1. $\underline{10} \times \underline{10} \times \underline{10} \times \underline{10} = 10{,}000$

2. $\underline{10} \times \underline{9} \times \underline{9} \times \underline{9} = 7{,}290$

3. $\underline{10} \times \underline{9} \times \underline{8} \times \underline{7} = 5{,}040$

DEFINITION Permutation of *n* Objects Taken *r* at a Time

A permutation of a set of n distinct objects taken r at a time without repetition is an arrangement of r of the n objects in a specific order.

THEOREM 2 Number of Permutations of *n* Objects Taken *r* at a Time

The number of permutations of n distinct objects taken r at a time without repetition is given by*

$$ {}_nP_r = n(n-1)(n-2)\cdot\cdots\cdot(n-r+1) \qquad r \text{ factors} $$

$$ {}_5P_2 = 5\cdot 4 \text{ factors} $$

or

$$ {}_nP_r = \frac{n!}{(n-r)!} \qquad 0 \le r \le n \qquad {}_5P_2 = \frac{5!}{(5-2)!} = \frac{5!}{3!} $$

Note: ${}_nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$ permutations of n objects taken n at a time.

Remember, by definition, $0! = 1$.

* In place of the symbol ${}_nP_r$, the symbols P_r^n, $P_{n,r}$, and $P(n, r)$ are often used.

FACTORIAL: $n! = n(n-1)(n-2)\cdots(2)(1)$

(ONLY DEFINED FOR NON-NEG. INTEGERS

$0! = 1$ (CONVENTION))

→ $n!$: NUMBER OF WAYS TO ARRANGE / ORDER n OBJECTS

$$ {}_nP_r = n(n-1)\cdots(n-r+1) = \frac{n(n-1)\cdots(n-r+1)\cancel{(n-r)}\cdots\cancel{(1)}}{\cancel{(n-r)}\cdots\cancel{(1)}} $$

$$ = \frac{n!}{(n-r)!} $$

Example 2.

A musician has written and recorded 22 songs. If she want to release 10 of these songs as an album (order matters!), how many possible ways are there for her to do this?

$$\underset{1}{\underline{22}}\quad \underset{2}{\underline{21}}\quad \underset{3}{\underline{20}}\quad \underset{4}{\underline{19}}\quad \underset{5}{\underline{18}}\quad \underset{6}{\underline{17}}\quad \underset{7}{\underline{16}}\quad \underset{8}{\underline{15}}\quad \underset{9}{\underline{14}}\quad \underset{10}{\underline{13}}$$

$${}_{22}P_{10} = \frac{22!}{(22-10)!} = \frac{22!}{12!} = \frac{22\cdot 21 \cdots 13 \cdot \cancel{12}\cdot \cancel{11} \cdots \cancel{1}}{\cancel{12}\cdot \cancel{11} \cdots \cancel{1}}$$

$$\approx 2.3465 \times 10^{12}$$

Example 3.

A musician has written a song with the following structure:

Verse, Chorus, Verse, Chorus, Bridge, Verse, Chorus.

Now a lyricists has written 5 different verses, 2 different choruses, and 3 different bridges. If each verse must be different, and each chorus must be the same, how many songs can be created from these lyrics with the given structure?

$$\underbrace{\underline{{}_5P_3}}\qquad \underline{2} \qquad \underline{3}$$

CHOOSE 3 VERSES FROM 5, ORDER MATTERS

CHOOSE 1 CHORUS FROM 2

CHOOSE 1 BRIDGE FROM 3

PERMUTATION KEYWORD

ORDER, ARRANGE

$$\underset{1}{\underline{5}} \times \underset{2}{\underline{4}} \times \underset{3}{\underline{3}}$$

$$60 \times 2 \times 3 = \boxed{360}$$

DEFINITION Combination of n Objects Taken r at a Time

A **combination** of a set of n distinct objects taken r at a time without repetition is an r-element subset of the set of n objects. The arrangement of the elements in the subset does not matter.

ex. 5 CARD POKER HAND, DEALT FROM DECK OF 52

Example 4.

How many ways are there to choose 3 aces from a standard deck of 52 cards if the order doesn't matter?

4 ACES ACE OF HEARTS ♡ H
SPADES ♤ S
DIAMONDS ◇ D
CLUBS ♧ C

3!

$_4C_3$

SCH, SHC, CSH, CHS, HSC, HCS,

CHD, CDH, HCD, HDC, DCH, DHC,

HDS, HSD, DHS, DSH, SHD, SDH,

DSC, DCS, SDC, SCD, CDS, CSD.

4 "COMBINATIONS"

IF ORDER MATTERED, # WAYS WOULD BE $_4P_3 = 24$

GIVEN 3 CARDS, # WAYS TO ARRANGE/ORDER/PERMUTE THEM

IS $_3P_3 = \frac{3!}{(3-3)!} = \frac{3!}{0!} = 3 \cdot 2 \cdot 1 = 6$

ORDER DOES NOT MATTER

$$_4C_3 = \frac{_4P_3}{3!} = 4$$

THEOREM 3 Number of Combinations of n Objects Taken r at a Time

The number of combinations of n distinct objects taken r at a time without repetition is given by*

$$ {}_nC_r = \binom{n}{r} = \frac{{}_nP_r}{r!} = \frac{n!}{r!(n-r)!} \quad 0 \le r \le n \qquad {}_{52}C_5 = \binom{52}{5} = \frac{{}_{52}P_5}{5!} = \frac{52!}{5!(52-5)!} $$

* In place of the symbols ${}_nC_r$ and $\binom{n}{r}$, the symbols C_r^n, $C_{n,r}$ and $C(n, r)$ are often used.

Example 5.

If you flip a coin 20 times, you get a sequence of heads (H) and tails (T).

1. How many different sequences of heads and tails are possible?
2. How many different sequences of heads and tails have exactly 8 heads?

H, T

1.

$$\underset{\substack{H\\T}}{\underline{2}} \times \underset{\substack{H\\T}}{\underline{2}} \times \underset{\substack{H\\T}}{\underline{2}} \times \underline{\quad} \cdots \underset{\substack{H\\T}}{\underline{2}} \times \underset{\substack{H\\T}}{\underline{2}} = 2^{20} = \boxed{1,048,576}$$

2. _ _ H _ _ H _ H H H

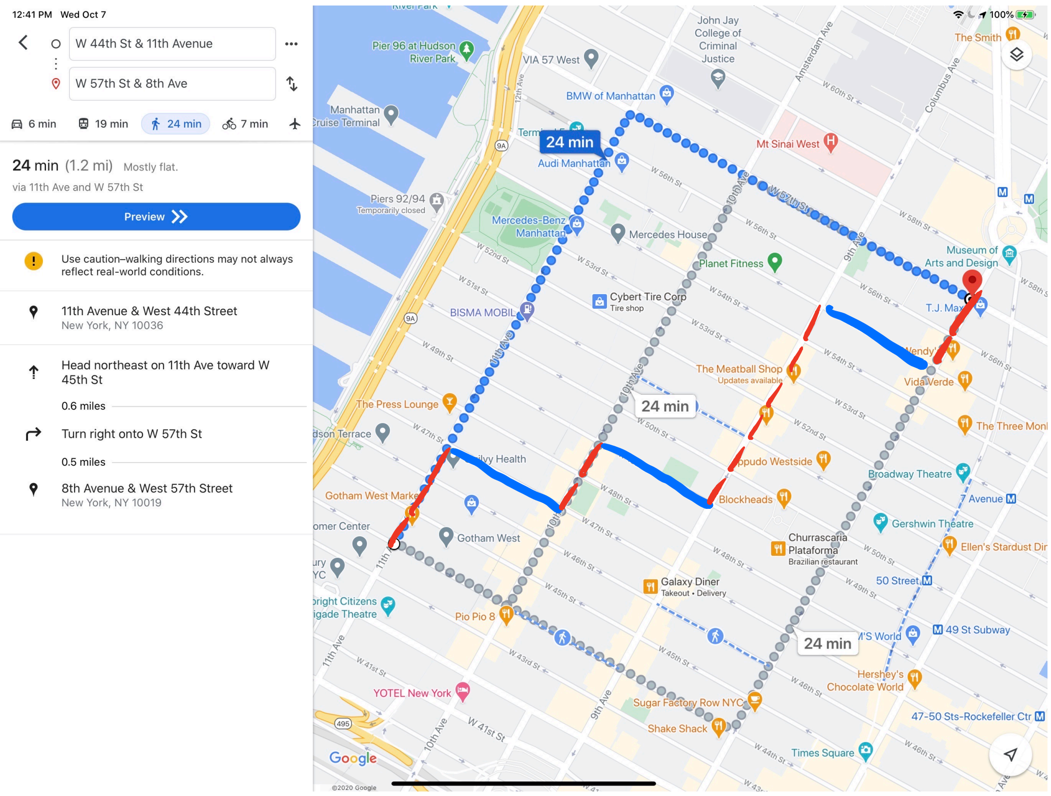
_ _ _ H H _ _ _ _ H

(1) CHOOSE THE 8 POSITIONS FOR H $\quad {}_{20}C_8 = \boxed{125,970}$

(2) ALL THE REST ARE T. $\quad 1$

Example 6.

How many ways are there to walk from corner of W 44 St and 11th Ave to the corner of W 57 St and 8th Ave, assuming you do not go out of your way (only walk north and east)?



"THE TAXI CAB PROBLEM"

NORTH 13 BLOCKS, EAST 3 BLOCKS

⇒ WALK 16 BLOCKS

_ _ _ E _ _ E _ _ _ _ _ _ E _ _

N N N E N N E N N N N N N E N N

16 BLOCKS, CHOOSE 3 TO GO EAST

↳ $_{16}C_3 = 560$

16 BLOCKS, CHOOSE 13 TO GO NORTH

$_{16}C_{13} = 560$

Note: $_nC_r = \frac{n!}{r!(n-r)!}$

$_nC_{(n-r)} = \frac{n!}{(n-r)![n-(n-r)]!} = \frac{n!}{(n-r)!r!}$ SAME

Example 7.

The fish section of a pet store is stocked with 8 guppies, 6 angelfish, 13 goldfish, and 9 rainbowfish.

1. How many ways are there to select one of each?
2. How many ways are there to select two of each?
3. How many ways are there to select 7 fish, if you must select at least one of each?

1. $\underset{\text{GUP}}{\underline{8}} \times \underset{\text{ANG.}}{\underline{6}} \times \underset{\text{GF}}{\underline{13}} \times \underset{\text{RF}}{\underline{9}} = \boxed{5616}$

2. $\underset{\text{GUP}}{\underline{{}_8C_2}} \times \underset{\text{ANG.}}{\underline{{}_6C_2}} \times \underset{\text{GF}}{\underline{{}_{13}C_2}} \times \underset{\text{RF}}{\underline{{}_9C_2}} = \boxed{1{,}179{,}360}$

3. $\underline{8} \times \underline{6} \times \underline{13} \times \underline{9} \times (\underline{4},\ \underline{9}\ \underline{6}\ \underline{0})$

(1) PICK 1 GUPPY $\quad {}_8C_1 = 8$

(2) PICK 1 ANG $\quad {}_6C_1 = 6$

(3) PICK 1 GF $\quad {}_{13}C_1 = 13$

(4) PICK 1 RF $\quad {}_9C_1 = 9$

(5) PICK 3 FISH OF ANY TYPE, $\quad {}_{32}C_3 = 4960$

32 FISH REMAIN

27,855,360

Example 9.

Suppose you want to give each of your 4 nieces/nephews a Christmas card containing some money. You have ten (identical) $20 bills that you want to distribute between the 4 (distinct) envelopes so that no envelope contains $0. How many ways are there to do this?

https://en.wikipedia.org/wiki/Stars_and_bars_(combinatorics)

CHOOSE 3 OUT OF THE 9 GAPS BETWEEN STARS IN WHICH TO PUT BARS.

$$ {}_9C_3 = \boxed{84} $$