8.5 Probability Distributions and Expected Value

Example/Discussion Problems

Random Variable

A **random variable** is a function that assigns a real number to each outcome of an experiment.

1. An experiment consists of rolling two fair 6-sided dice.
   - The first die has its faces labeled 1, 3, 3, 5, 5, 5.
   - The second die has its faces labeled 2, 2, 2, 4, 4, 6.

   Define the random varible X and Y as follows.
   - x = sum of faces rolled
   - y = non-negative difference of faces rolled

   For each random variable, create a table that lists its possible values and the corresponding probabilities that the random variable equals these values.

x	
$p(x)$	

y	
$p(y)$	

Probability Distribution A table that lists the possible values of a random variable, together with the corresponding probabilities, is called a **probability distribution**.

Note 1. If a random variable has n possible values $x_1, x_2, \ldots, x_n$ then

$$p(x_1) + p(x_2) + \ldots + p(x_n) = 1.$$

Note 2. When a random variable has only a finite number of possble values, a **probability distribution** is a function that assigns a probability to every possible value of the random variable. In this case, a probability distribution is also called a **probability distribution function**, or simply a **probability function**.

2. Three batteries are randomly selected from a drawer that contains 12 new batteries and 6 old batteries. Define the random variable x to be the number of new batteries selected. Describe the probability distribution with a table and a histogram. Round all probabilities to 4 decimal places.

3. Suppose the experiment in example 2 is repeated 10,000 times. How many times would you expect to select $x = 0$ new batteries? $x = 1$? $x = 2$? $x = 3$? Using these values, what would be the average number of new batteries selected over all 10,000 repititions of the experiment?

Expected Value

Suppose the random variable x can take on the n values $x_1, x_2, x_3, \ldots, x_n$. Also, suppose the probabilities that these values occur are, respectively, $p_1, p_2, p_3, \ldots, p_n$. Then the **expected value** of the random variable is

$$E(x) = x_1p_1 + x_2p_2 + x_3p_3 + \cdots + x_np_n.$$

4. A school raises money by selling 2,500 raffle tickets for \$10 each. After selling all of the tickets, 6 tickets ae chosen randomly to receive prizes: 3 tickets win \$500 each, 2 tickets win \$1,000 each, and one tickets wins \$5,000. Define the random variable x to be the amount of money won/lost by purchasing one raffle ticket.

$$x = \text{money in} - \text{money out}$$

What is the expected value of x?

5. In any given calendar year, a factory worker has a 0.6% chance of becoming disabled on the job and unable to work. Thus, a labor union offers factory workers a 1-year disability isnsurance policy such that workers who purchase the policy and experience a workplace disability receive a one-time payment of \$250,000. If the labor union simply wants to break even, how much should they charge for this policy?

6. Suppose 68% of americans own a car. If 3 american are randomly selected and x is the number that own a car, find the expected value for x.

Expected Value for Binomial Probability

For binomial probability, $E(x) = np$. In other words, the expected number of successes is the number of trials times the probability of success in each trial.

Random Variable

A **random variable** is a function that assigns a real number to each outcome of an experiment.

VALUE IS UNKNOWN UNTIL THE EXPERIMENT IS PERFORMED.

1. An experiment consists of rolling two fair 6-sided dice.

   - The first die has its faces labeled 1, 3, 3, 5, 5, 5.
   - The second die has its faces labeled 2, 2, 2, 4, 4, 6.

   Define the random varible X and Y as follows.

   - x = sum of faces rolled
   - y = non-negative difference of faces rolled

   For each random variable, create a table that lists its possible values and the corresponding probabilities that the random variable equals these values.

x	3	5	7	9	11
$p(x)$	$3/36$	$8/36$	$14/36$	$8/36$	$3/36$

ADD UP TO 1 (100%)

y	1	3	5
$p(y)$	$22/36$	$13/36$	$1/36$

ADD UP TO 1 (100%)

X	1	3	3	5	5	5
2	3	5	5	7	7	7
2	3	5	5	7	7	7
2	3	5	5	7	7	7
4	5	7	7	9	9	9
4	5	7	7	9	9	9
6	7	9	9	11	11	11

36 POSSIBLE OUTCOMES

Y	1	3	3	5	5	5
2	1	1	1	3	3	3
2	1	1	1	3	3	3
2	1	1	1	3	3	3
4	3	1	1	1	1	1
4	3	1	1	1	1	1
6	5	3	3	1	1	1

36 POSSIBLE OUTCOMES

Probability Distribution A table that lists the possible values of a random variable, together with the corresponding probabilities, is called a **probability distribution**.

Note 1. If a random variable has n possible values $x_1, x_2, \ldots, x_n$ then

$$p(x_1) + p(x_2) + \ldots + p(x_n) = 1.$$

Note 2. When a random variable has only a finite number of possble values, a **probability distribution** is a function that assigns a probability to every possible value of the random variable. In this case, a probability distribution is also called a **probability distribution function**, or simply a **probability function**.

2. Three batteries are randomly selected from a drawer that contains 12 new batteries and 6 old batteries. Define the random variable x to be the number of new batteries selected. Describe the probability distribution with a table and a histogram. Round all probabilities to 4 decimal places.

(HYPERGEOMETRIC DISTRIBUTION   hypergeometric distribution n=3, m=12, N=18)

x	0	1	2	3
$p(x)$	$\frac{C(12,0)C(6,3)}{C(18,3)}$	$\frac{C(12,1)C(6,2)}{C(18,3)}$	$\frac{C(12,2)C(6,1)}{C(18,3)}$	$\frac{C(12,3)C(6,0)}{C(18,3)}$
	.0245	.2206	.4853	.2696

← POSSIBLE VALUES

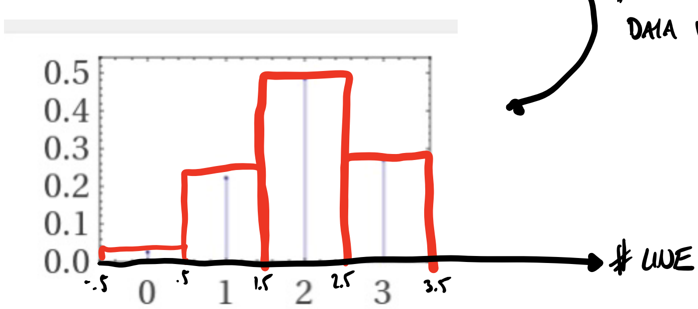
PROBABILITY OF SELECTING 0 OUT OF 12 NEW BATT. & 3 OUT OF 6 OLD BATT. FROM A TOTAL OF 18 BATT.

$P(A) = \frac{n(A)}{n(S)}$

PROBABILITY OF SELECTING 2 OUT OF 12 NEW BATT. & 1 OUT OF 6 OLD BATT. FROM A TOTAL OF 18 BATT.

$P(A) = \frac{n(A)}{n(S)}$

A HISTOGRAM IS A BAR CHART WHERE HORIZONTAL AXIS IS A # LINE



"DATA VISUALIZATION"

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BARS HAVE WIDTH 1 (DEFAULT.)

3. Suppose the experiment in example 2 is repeated 10,000 times. How many times would you expect to select $x = 0$ new batteries? $x = 1$? $x = 2$? $x = 3$? Using these values, what would be the average number of new batteries selected over all 10,000 repitions of the experiment?

x	0	1	2	3
$p(x)$	$\frac{C(12,0)C(6,3)}{C(18,3)}$	$\frac{C(12,1)C(6,2)}{C(18,3)}$	$\frac{C(12,2)C(6,1)}{C(18,3)}$	$\frac{C(12,3)C(6,0)}{C(18,3)}$
	.0245	.2206	.4853	.2696

← ADD UP TO 1.

$\frac{245}{10{,}000}$

IF WE REPEAT THIS EXPERIMENT 10,000 TIMES & RECORD 10,000 VALUES FOR X

$\underbrace{0, 2, \ldots, 1.}_{\text{LIST OF 10,000 VALUES.}}$

I EXPECT MY LIST OF 10,000 VALUES TO CONTAIN...

245 0's   2206 1's   4853 2's   2696 3's

AVERAGE VALUE OF X

$$\frac{245 \times 0 + 2206 \times 1 + 4853 \times 2 + 2696 \times 3}{10{,}000}$$

$$\frac{245 \times 0}{10{,}000} + \frac{2206 \times 1}{10{,}000} + \frac{4853 \times 2}{10{,}000} + \frac{2696 \times 3}{10{,}000}$$

$$\frac{245}{10{,}000} \times 0 + \frac{2206}{10{,}000} \times 1 + \frac{4853}{10{,}000} \times 2 + \frac{2696}{10{,}000} \times 3$$

$$(.0245)(0) + (.2206)(1) + (.4853)(2) + (.2696)(3) = 2$$

EXPECTED VALUE FOR RAN VAR. X

LONG TERM AVERAGE

Expected Value

Suppose the random variable x can take on the n values $x_1, x_2, x_3, \ldots, x_n$. Also, suppose the probabilities that these values occur are, respectively, $p_1, p_2, p_3, \ldots, p_n$. Then the **expected value** of the random variable is

$$E(x) = \underbrace{x_1p_1 + x_2p_2 + x_3p_3 + \cdots + x_np_n}_{\text{WEIGHTED AVERAGE OF } x_1, x_2, \ldots, x_n}.$$

4. A school raises money by selling 2,500 raffle tickets for \$10 each. After selling all of the tickets, 6 tickets ae chosen randomly to receive prizes: 3 tickets win \$500 each, 2 tickets win \$1,000 each, and one tickets wins \$5,000. Define the random variable x to be the amount of money won/lost by purchasing one raffle ticket.

$$x = \text{money in} - \text{money out}$$

What is the expected value of x?

EXPERIMENT: SELECT 6 TICKETS RANDOMLY FROM 2,500 TICKETS
- 3 WIN \$500
- 2 WIN \$1000
- 1 WIN \$5000

	0 - 10	500 - 10	1000 - 10	5000 - 10
x	-10	490	990	4990
$p(x)$	$\frac{2494}{2500}$	$\frac{3}{2500}$	$\frac{2}{2500}$	$\frac{1}{2500}$
	.9976	.0012	.0008	.0004

EXPECTED VALUE FOR X:

$$E(x) = (.9976)(-10) + (.0012)(490) + (.0008)(990) + (.0004)(4990)$$

$$= -6.6 \text{ DOLLARS.}$$

IF YOU WERE TO REPEAT THIS EXPERIMENT OVER & OVER, RECORDING THE VALUE OF X EACH TIME, THE AVERAGE OF ALL RECORDED VALUES WILL GET CLOSER & CLOSER TO -6.6 AS THE NUMBER OF REPITITIONS OF THE EXPERIMENT INCREASES.

$\frac{.6}{100} = .006$

5. In any given calendar year, a factory worker has a 0.6% chance of becoming disabled on the job and unable to work. Thus, a labor union offers factory workers a 1-year disability isnsurance policy such that workers who purchase the policy and experience a workplace disability receive a one-time payment of \$250,000. If the labor union simply wants to break even, how much should they charge for this policy?

EXPERIMENT: OBSERVE IF A WORKER (WHO PURCHASES INSURANCE) GETS IN AN ACCIDENT OVER THE CALENDAR YEAR.

LET X = MONEY GAINED/LOST BY UNION ON ONE POLICY.
= MONEY IN - MONEY OUT.

LET C : COST OF THE POLICY (PRICE)

C - 0

x	C	C - 250,000
$p(x)$	.994	.006

BREAK EVEN : EXPECTED VALUE = 0

AVERAGE VALUE

$$0 = E(x) = .994c + .006(c - 250{,}000) = 0$$

$$\underbrace{.994c + .006c}\ - 1500 = 0$$

$$c - 1500 = 0$$

$$\boxed{c = 1500}$$

CHARGE \$1500

6. Suppose 68% of americans own a car. If 3 american are randomly selected and x is the number that own a car, find the expected value for x.

n

NOTE: X IS # SUCCESSES IN 3 TRIALS, EACH WITH PROBABILITY $p = .68$ OF SUCCESS.

$n = 3$
$p = .68$
$q = .32$

BINOMIAL EXPERIMENT : $P(x) = C(n,x)\, p^x q^{n-x}$

OF SUCCESSES X IS A BINOMIAL RANDOM VARIABLE

① PROBABILITY DISTRIBUTION

x	0	1	2	3
$p(x)$	$C(3,0)(.68)^0(.32)^3$	$C(3,1)(.68)^1(.32)^2$	$C(3,2)(.68)^2(.32)^1$	$C(3,3)(.68)^3(.32)^0$
	.0328	.2089	.4439	.3144

$P(x=0)$

② EXPECTED VALUE

$$E(x) = (.0328)(0) + (.2089)(1) + (.4439)(2) + (.3144)(3) = \ldots$$

"SHORTCUT"

Expected Value for Binomial Probability

For binomial probability, $E(x) = np$. In other words, the expected number of successes is the number of trials times the probability of success in each trial.

68% OF TRIALS ARE SUCCESSES

$E(x) = .68 \times 3 = \boxed{2.04}$

$p \times n$