8.2 Combinations

Example/Discussion Problems

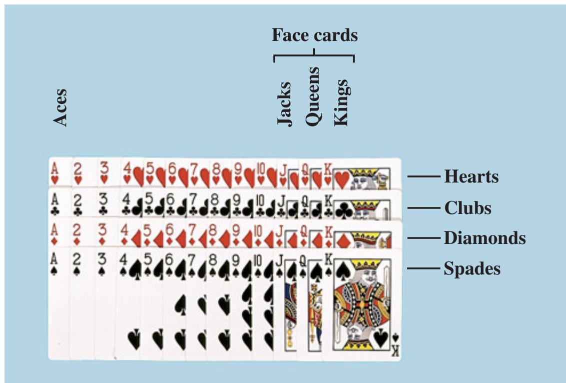


Figure 1: A standard deck of 52 cards.

1. How many 5-card poker hands are possible?

Combinations

If $C(n, r)$, denotes the number of combinations of n elements taken r at a time, where $r \le n$, then

$$C(n,r) = \frac{n!}{(n-r)!\,r!}.$$

2. How many 5-card poker hands are possible with 2 pairs?
3. How many 5-card poker hands are possible with 3 of a kind?
4. When playing 5-card poker, what is the probability of being dealt 2 pairs? 3 of a kind?
5. A sandwich shop instructs customers to place their order as follows.

Choose 1	Choose 2	Choose 0, 1, 2, or 3
White	Chicken	Lettuce
Whole Wheat	Beef	Tomatoe
Multi-grain	Bacon	Onions
	Black Beans	Pickles
	Red Beans	Olives
	Avacado	Mayonaise
		Mustard
		Ketchup

How many different sandwiches is it possible to order?

6. In how man ways can 4 coworkers each take 2 donuts from a box of 12 distinct donuts?

7. How many ways can a team of 16 choose 3 players to play offense, 2 players to play defense, and 1 player to play goalie?

   (Compare to choosing 1 president, 1 vice-president, 1 treasurer.)

8. A woman sending Christmas cards to her three nephews. Before mailing the 3 envelopes, she takes 9 \$100 bills and randomly splits them between the three envelopes so that each nephew receives at least one \$100 bill. How many ways are there for her to do this?

   Challenge: What if it is not required that each nephew receive at least one \$100 bill?

9. How many ways are there to walk from W 43 St and 11th Ave to W 57 St and 6th Ave, without walking out of the way and wihtout taking Broadway?

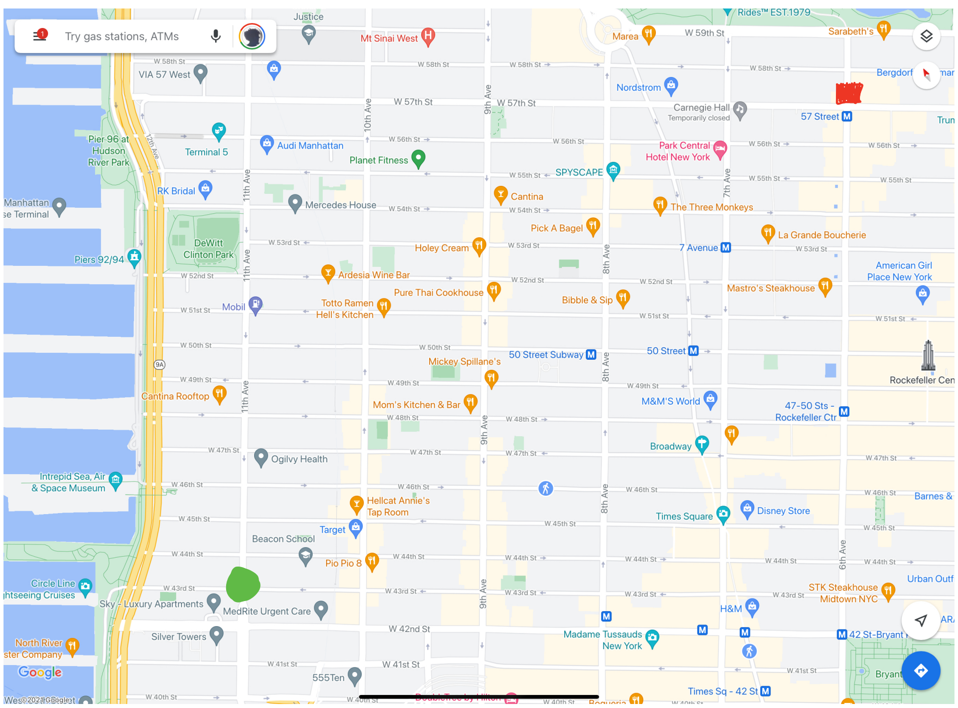


Figure 2: maps.google.com

10. How many ways are there to *drive* from W 43 St and 11th Ave to W 57 St and 6th Ave, without walking out of the way and wihtout taking Broadway?

Permutations	Combinations
Different orderings or arrangements of the r objects are different permutations.	Each choice or subset of r objects gives one combination. Order within the group of r objects does not matter.
$P(n, r) = \dfrac{n!}{(n-r)!}$	$C(n, r) = \dfrac{n!}{(n-r)!\,r!}$
Clue words: arrangement, schedule, order	Clue words: group, committee, set, sample
Order matters!	**Order does not matter!**

Figure 3: Permutations vs. combinations.

1. How many 5-card poker hands are possible?

$$P(n,r) = \frac{n!}{(n-r)!}$$

52 CARDS IN DECK :

1 POKER HAND	10♡	J♡	Q♡	K♡	A♡
A NOTHER?	J♡	10♡	Q♡	K♡	A♡

NO.

BOTH COUNTED AS DIFFERENT PERMUTATIONS (ORDER MATTERS)

\# CHOICES $\underset{1^{st}}{\underline{52}} \times \underset{2^{nd}}{\underline{51}} \times \underset{3^{rd}}{\underline{50}} \times \underset{4^{TH}}{\underline{49}} \times \underset{5^{TH}}{\underline{48}} = P(52,5) = 311{,}875{,}200$

IN POKER ORDER DOESN'T MATTER

$P(n,r)$ = # OF WAYS TO SELECT ~~AND ARRANGE~~ r OBJECTS FROM n OBJECTS (ORDER MATTERS!)

" # OF PERMUTATIONS OF r OBJECTS TAKEN FROM n OBJECTS "

LET $C(n,r)$ = # OF WAYS TO SELECT r OBJECTS FROM n OBJECTS SUCH THAT THE ORDER DOES NOT MATTER

"NUMBER OF COMBINATIONS OF r OBJECTS TAKEN FROM n OBJECTS"

1. How many 5-card poker hands are possible?

$\underset{5}{\underline{10♡}} \times \underset{4}{\underline{J♡}} \times \underset{3}{\underline{Q♡}} \times \underset{2}{\underline{K♡}} \times \underset{1}{\underline{A♡}} = 5!$

WAYS TO ORDER 5 CARDS.

IF ORDER MATTERED : $P(52,5)$

AND ONE WAY TO ARRIVE AT THIS NUMBER IS IN 2 STEPS:

$$\boxed{P(52,5) = \underbrace{C(52,5)}_{\text{CHOOSE 5 CARDS}} \times \underbrace{5!}_{\text{ARRANGE/ORDER 5 CHOSEN CARDS}}} \Rightarrow C(52,5) = \frac{P(52,5)}{5!}$$

$$\Rightarrow C(52,5) = \frac{\;\frac{52!}{(52-5)!}\;}{5!} = \frac{52!}{5!\,(52-5)!}$$

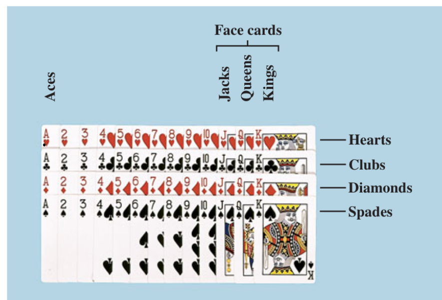
IN GENERAL:

$$\boxed{\begin{aligned} P(n,r) &= C(n,r) \times r! \\ C(n,r) &= \frac{P(n,r)}{r!} = \frac{n!}{r!\,(n-r)!} \end{aligned}}$$

2. How many 5-card poker hands are possible with 2 pairs?

3. How many 5-card poker hands are possible with 3 of a kind?

2. 2 PAIRS: 2 CARDS OF ONE DENOMINATION
VALUE
2 CARDS OF ANOTHER DENOMINATION
1 CARD OF A THIRD DENOMINATION



Multiplication Principle

Suppose n choices must be made, with

m_1 ways to make choice 1,

m_2 ways to make choice 2,

and so on, with

m_n ways to make choice n.

Then there are

$$m_1 \cdot m_2 \cdot \cdots \cdot m_n$$

different ways to make the entire sequence of choices.

·) Q♡ Q♢ 8♧ 8♤ 5♢

\# WAYS TO GET 2 PAIRS

44 CARDS
NOT Q
NOT 8

$$\underbrace{\underset{\text{CHOOSE VALUE FOR 1ST PAIR}}{C(13,1)} \times \underset{\text{CHOOSE CARDS FOR 1ST PAIR}}{C(4,2)}}_{\text{1ST PAIR}} \times \underbrace{\underset{\text{CHOOSE VALUE FOR 2ND PAIR}}{C(12,1)} \times \underset{\text{CHOOSE CARDS FOR 2ND PAIR}}{C(4,2)}}_{\text{2ND PAIR}} \times \underset{\text{CHOOSE 5TH CARD}}{C(44,1)}$$

$$= 247{,}104$$

NOTE: $C(n,1) = \dfrac{n!}{1!\,(n-1)!} = \dfrac{n \cdot \cancel{(n-1)!}}{\cancel{(n-1)!}} = n$

5. A sandwich shop instructs customers to place their order as follows.

NOTE: $C(n,0) = \dfrac{n!}{0!\,(n-0)!} = \dfrac{n!}{n!} = 1$

Choose 1	Choose 2	Choose 0, 1, 2, or 3
White	Chicken	Lettuce
Whole Wheat	Beef	Tomatoe
Multi-grain	Bacon	Onions
	Black Beans	Pickles
	Red Beans	Olives
	Avacado	Mayonaise
		Mustard
		Ketchup

CHOOSE 0 ITEMS FROM COL 3 : $\underbrace{C(3,1)}_{\text{COL 1}} \times \underbrace{C(6,2)}_{\text{COL 2}} \times \underbrace{C(8,0)}_{\text{COL 3}}$

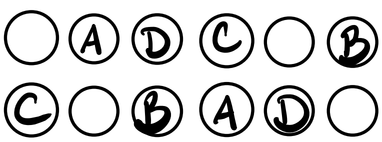
CHOOSE 1 ITEM FROM COL 3 : $\underbrace{C(3,1)}_{\text{COL 1}} \times \underbrace{C(6,2)}_{\text{COL 2}} \times \underbrace{C(8,1)}_{\text{COL 3}}$

CHOOSE 2 ITEMS FROM COL 3 : $\underbrace{C(3,1)}_{\text{COL 1}} \times \underbrace{C(6,2)}_{\text{COL 2}} \times \underbrace{C(8,2)}_{\text{COL 3}}$

CHOOSE 3 ITEMS FROM COL 3 : $\underbrace{C(3,1)}_{\text{COL 1}} \times \underbrace{C(6,2)}_{\text{COL 2}} \times \underbrace{C(8,3)}_{\text{COL 3}}$

ADD TOGETHER FOR TOTAL.

6. In how man ways can 4 coworkers each take 2 donuts from a box of 12 distinct donuts?



COWORKERS A, B, C, D.

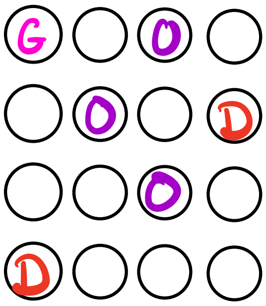
MULT. PRINC.

$\underbrace{C(12,2)}_{\text{CHOOSE DONUTS FOR COWORKER A}} \times \underbrace{C(10,2)}_{\text{CHOOSE DONUTS FOR COWORKER B}} \times \underbrace{C(8,2)}_{\text{CHOOSE DONUTS FOR COWORKER C}} \times \underbrace{C(6,2)}_{\text{CHOOSE DONUTS FOR COWORKER D}}$

$\dfrac{12!}{2!\,\cancel{10!}} \times \dfrac{\cancel{10!}}{2!\,\cancel{8!}} \times \dfrac{\cancel{8!}}{2!\,\cancel{6!}} \times \dfrac{\cancel{6!}}{2!\,4!} = \dfrac{12!}{2!\,2!\,2!\,4!} = 1{,}247{,}400$

7. How many ways can a team of 16 choose 3 players to play offense, 2 players to play defense, and 1 player to play goalie?

(Compare to choosing 1 president, 1 vice-president, 1 treasurer.)



CHOOSE 3 OFFENSIVE PLAYERS FROM 16 PLAYERS : $C(16,3)$

CHOOSE 2 DEFENSIVE PLAYERS FROM 13 PLAYERS : $C(13,2)$

CHOOSE 1 GOALIE FROM 11 PLAYERS : $C(11,1)$

MULT. PRINCIPLE : $C(16,3) \times C(13,2) \times C(11,1) = \boxed{480,480}$

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8. A woman sending Christmas cards to her three nephews. Before mailing the 3 envelopes, she takes 9 \$100 bills and randomly splits them between the three envelopes so that each nephew receives at least one \$100 bill. How many ways are there for her to do this?

"STARS & BARS"
(COUNTING METHOD)

★ ★ | ★ ★ ★ ★ | ★ ★ ★

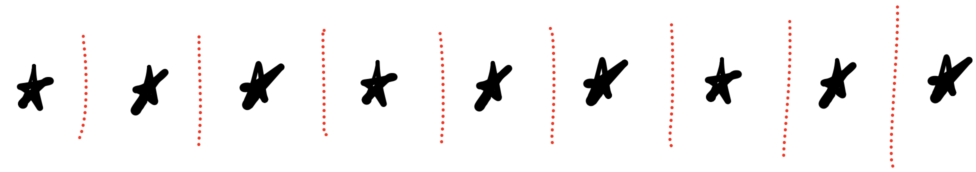
#1 #2 #3

★ | ★ ★ ★ ★ ★ ★ ★ | ★

#1 #2 #3

★ ★ ★ ★ | ★ ★ | ★ ★ ★

#1 #2 #3



How many ways are there to insert 2 bars in between 9 stars?
How many ways are there to choose 2 places to put bars, out of 8 places?

$$C(8,2) = \frac{8!}{2!6!} = \frac{8\cdot 7}{2} = 28$$